

ON THE FORMATION OF SHOCKS FOR QUASILINEAR WAVE EQUATIONS

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ABSTRACT. The paper is devoted to the study of shock formation of the 3-dimensional quasilinear wave equation

$$-(1 + 3G''(0)(\partial_t \phi)^2)\partial_t^2 \phi + \Delta \phi = 0, \quad (\star)$$

where $G''(0)$ is a non-zero constant. We will exhibit a family of smooth initial data and show that the foliation of the incoming characteristic hypersurfaces collapses. Similar to 1-dimensional conservational laws, we refer this specific type breakdown of smooth solutions as shock formation. Since (\star) satisfies the classical null condition, it admits global smooth solutions for small data. Therefore, we will work with large data (in energy norm). Moreover, no symmetry condition is imposed on the initial datum.

We emphasize the geometric perspectives of shock formations in the proof. More specifically, the key idea is to study the interplay between the following two objects:

- (1) the energy estimates of the linearized equations of (\star) ;
- (2) the differential geometry of the Lorentzian metric $g = -\frac{1}{(1 + 3G''(0)(\partial_t \phi)^2)}dt^2 + dx_1^2 + dx_2^2 + dx_3^2$.

Indeed, the study of the characteristic hypersurfaces (implies shock formation) is the study of the null hypersurfaces of g .

The techniques in the proof are inspired by the work [5] in which the formation of shocks for 3-dimensional relativistic compressible Euler equations with small initial data is established. We also use the short pulse method which is introduced in the study of formation of black holes in general relativity in [6] and generalized in [13].

1. INTRODUCTION

This paper is devoted to the study of the following quasilinear wave equation

$$-(1 + 3G''(0)(\partial_t \phi)^2)\partial_t^2 \phi + \Delta \phi = 0, \quad (1.1)$$

where $G''(0)$ is a *nonzero* constant and $\phi \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^3; \mathbb{R})$ is a smooth solution. We propose a geometric mechanism for shock formations, i.e. how the smoothness of ϕ breaks down. We remark that if $G''(0) = 0$, the equation is linear so that no shock is expected. We will see that (\star) can be regarded as the simplest quasilinear wave equations that can be derived from the least action principle. The equation can also be regarded as a model equation for the nonlinear version of Maxwell equations in nonlinear electromagnetic theory, in which the shocks can be observed experimentally. The shock formation in nonlinear electromagnetic theory will be the subject of a forthcoming paper by the authors.

The breakdown mechanism is a central object in the theory of quasilinear hyperbolic equations. We give a brief account on the results related to the current work. In [2], Alinhac proved a conjecture of Hörmander concerning upper bounds of the lifespan for the solutions of $-\partial_t^2 \phi + \Delta \phi = \partial_t \phi \partial_t^2 \phi$ on \mathbb{R}^{2+1} . This equation was first introduced by F. John (see the survey paper [11] and the references therein). He [10] studied the rotationally symmetric cases and obtained upper bounds for the lifespan of the solutions. In [2] and [3], without any symmetry assumptions, Alinhac not only shows the solution blows up but also gives a very precise description of the solution near the blow-up point. Despite the slight different forms

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of the equations, Alinhac's results are fundamentally different from the current work in the following aspects: (1) He deals with small data problem. We will (and have to) deal with large data problem. (2) He uses Nash-Moser method to recover the loss of derivatives. Based on the variational nature of (\star) , we can close the energy estimates with finite many derivatives. (3) Just as Alinhac's work, we can give a detailed account on the behaviors of the solutions near blow-up points. Moreover, we show that the singularities formed are indeed shocks. (4) We can give a pure geometric interpretation of the shock formation (in terms of certain curvature tensors) and show that the blow-up behavior indeed can be read off from the initial data directly.

A major breakthrough in understanding the shock formations for the Euler equations has been made by D. Christodoulou in his monograph [5]. He considers the relativistic Euler equations for a perfect irrotational fluid with an arbitrary equation of state. Provided certain smallness assumptions on the initial data, he obtained a complete picture of shock formations in *three* dimensions. A similar result for classical Euler's equations has also been obtained by Christodoulou and S. Miao in [8]. The approaches are based on differential geometric methods originally introduced by Christodoulou and Klainerman in their monumental proof [7] of the nonlinear stability of the Minkowski spacetime in general relativity. Most recently, based on similar ideas, G. Holzegel, S. Klainerman, J. Speck and W. Wong have obtained remarkable results in understanding the stable mechanism for shock formations for certain types of quasilinear wave equations with small data in three dimensions, see their overview paper [9] and Speck's detailed proof [14]. We remark that one of the key ideas in [5] and [8] is to explore the variational structure of Euler's equations. This idea also plays a key rôle in the current work. We emphasize that [5] and [8] obtained sharp lower and upper bounds for the lifespan of smooth solutions associated to the given data *without* any symmetry conditions. Prior to [5] and [8], most of works on shock waves in fluid are limited to the simplified case of with spherical symmetry assumptions, i.e. essentially the one space dimension case. As an example, we mention [1] of Alinhac which studies the singularity formations for the compressible Euler equations on \mathbb{R}^2 with rotational symmetry.

All the aforementioned works have the common feature that the initial data are assumed to be small. However, since the nonlinearity in (\star) is cubic. By the classical result of Klainerman [12], for small smooth initial data, the solutions of (\star) are globally regular. In particular, we do not expect shock formation. We will use a special family of large data, so called *short pulse data*, in the current work. It was firstly introduced by D. Christodoulou in a milestone work [6] in understanding the formation of black holes in general relativity. By identifying an open set of initial data without any symmetry assumptions (the short pulse ansatz!), he shows that a trapped surface can form, even in vacuum space-time, from completely dispersed initial configurations and by means of the focusing effect of gravitational waves. Although the data are no longer close to Minkowski data, in other words, the data are no longer small, he is still able to prove a long time existence result for these data. This establishes the first result on the long time dynamics in general relativity and paves the way for many new developments on dynamical problems related to black holes. Shortly after Christodoulou's work, Klainerman and Rodnianski extends and significantly simplifies Christodoulou's work, see [13]. From a pure PDE perspective, the data appeared in the above works are carefully chosen large profiles which can be preserved by the Einstein equations along the evolution. The data in the current work are inspired by these ideas, in particular the idea in [6]. The initial profiles are designed in such a way that the shape of the data will be preserved along the evolution of (\star) .

As a summary, the data used in the paper are motivated by Christodoulou's work [6] and Klainerman-Rodnianski [13] on the formation of black holes in general relativity. The ideas of the proof are motivated by Christodoulou's work [5] and Christodoulou-Miao [8]. We have to overcome all the technical difficulties in the works mentioned above, in particular those in Christodoulou's works. At the same time, we would like to present a clearer geometric picture of the underlying shock formation mechanism.

1.1. The heuristics for shock formations. We rewrite (\star) in the so called *geometric form*:

$$-\frac{1}{c^2}\partial_t^2\phi + \Delta\phi = 0 \quad \dots\dots(\star_g),$$

where $c = (1 + 3G''(0)(\partial_t\phi)^2)^{-\frac{1}{2}}$. Recall that, if c was a constant, (\star_g) would describe the propagation of light in Minkowski space and c was the speed of light. In the current situation, we still regard c (which is **not** a constant) as the speed of light. But the speed of light depends on the position (t, x) in spacetime and the solution ϕ . This is of course the quasilinear nature of the equation. We now briefly review on the basics of shock formations for the inviscid Burgers' equation. The idea is to get a heuristic argument for the main equation (\star) and to motivate the main theorem.

The inviscid Burgers' equation can be written as

$$\partial_t u + u\partial_x u = 0 \quad \dots\dots(*).$$

We assume that $u \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x; \mathbb{R})$ is a smooth solution. Given smooth initial datum $u(0, x)$ (non-zero everywhere for simplicity), $(*)$ can be solved by the method of characteristics. A characteristic is a curve in $\mathbb{R}_{t,x}^2$ defined by the solution u . In the case of Burgers' equation, a characteristic is a straight line and it is determined by the initial datum $u(0, x)$ as follows: it is the unique line passing through $(0, x)$ with slope $\frac{1}{u(0, x)}$. The method of characteristics says that u is constant along each of the characteristics.

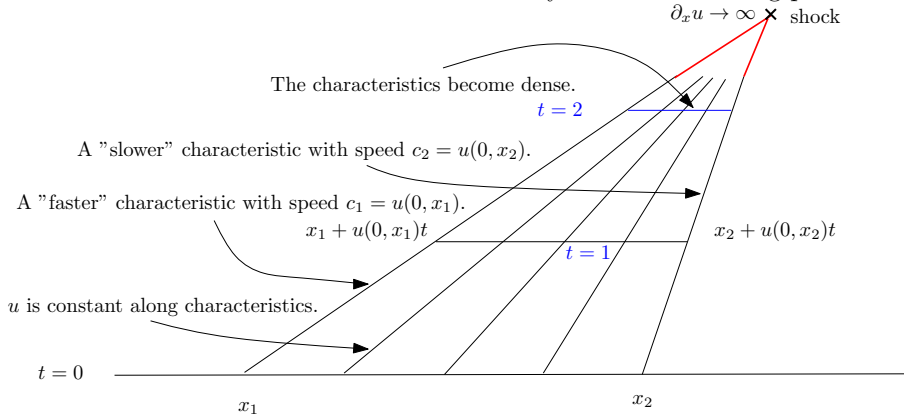
To make connections to the geometric form (\star_g) of the main equation (\star) , we also propose a *geometric form* of the Burgers' equation (we assume that $u \neq 0$ to make the following computation legitimate):

$$\frac{1}{c}\partial_t u + \partial_x u = 0 \quad \dots\dots(\star_g),$$

where $c = u$. Then c is the speed of the characteristics and it depends on the solution u .

We now consider two specific characteristics passing through x_1 and x_2 ($x_1 < x_2$). If we choose datum in such a way that $u(0, x_1) > u(0, x_2) > 0$, both the characteristics travel towards the right. Moreover, the characteristic on the left (noted as C_1) travels with speed $c_1 = u(0, x_1)$ and the characteristic on the right (noted as C_2) travels with speed $c_2 = u(0, x_2)$. Since C_1 travels faster than C_2 , C_1 will eventually catch up with C_2 . The collision of two characteristics causes the breakdown on the smoothness of the solution. In summary, we have a geometric perspective on shock formation: a "faster" characteristic catches up a "slower" one so that it causes a collapse of characteristics.

The above discussion can also be read off easily from the following picture:



In reality, instead of showing that characteristics collapse (which is on the heuristic level), we show that $|\partial_x u|$ blows up. Instead of being naively a derivative, $|\partial_x u|$ have an important *geometric interpretation*. Recall that the level sets of u are exactly the characteristics and the (t, x) -plane is foliated by the

characteristics (see the above picture). Therefore, $|\partial_x u|$ is the *density of the foliation by the characteristics*. As a consequence, we can regard the shock formation as the following geometric picture: *the foliation of characteristics becomes infinitely dense*.

We also recall a standard way to prove the blow-up of $|\partial_x u|$. The remarkable feature of this standard proof is that in three dimensions similar phenomenon happens for the main equation (\star) . Let $\underline{L} = \partial_t + u\partial_x$ be the generator vectorfield of the characteristics (for (\star) , the corresponding vectorfield are generators of null geodesics on the characteristic hypersurfaces). Therefore, by taking ∂_x derivatives, we obtain

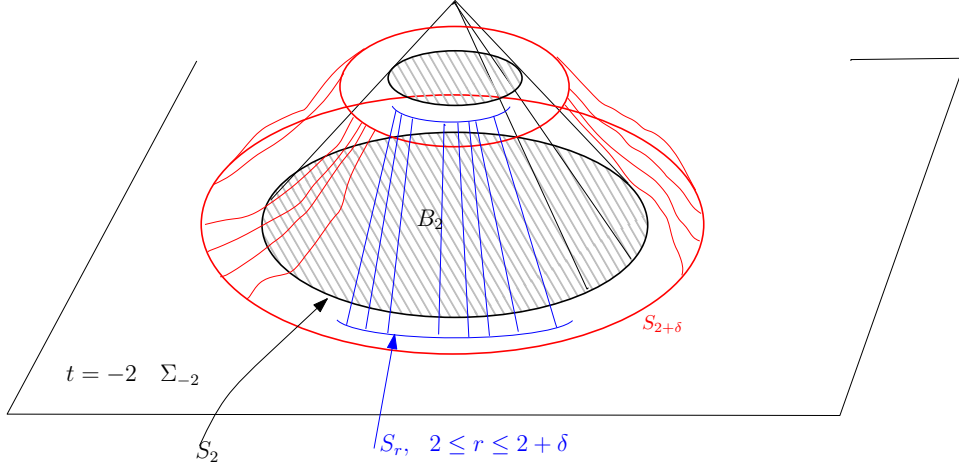
$$\underline{L}\partial_x u + (\partial_x u)^2 = 0.$$

This is a Riccati equation for $\partial_x u$ and the blow-up theory for $\partial_x u$ is standard. However, we would like to understand the blow-up in another way (which is intimately tied to the shock formation for (\star)). We define the *inverse density function* $\mu = -(\partial_x u)^{-1}$, therefore, along each characteristic curve, μ satisfies the following equation:

$$\underline{L}\mu(t, x) = -1,$$

i.e. $\underline{L}\mu$ is constant along each characteristic so that it is determined by its initial value. Therefore, μ will eventually become 0 which implies that the foliation becomes infinitely dense (For (\star) , we will also define an inverse density function μ for the foliation of characteristic hypersurfaces and show that $\underline{L}\mu(t, x)$ is almost a constant along each generating geodesic of the characteristic hypersurfaces).

We return to the main equation in the geometric form $-\frac{1}{c^2}\partial_t^2\phi + \Delta\phi = 0$ with $c = (1 + 3G''(0)(\partial_t\phi)^2)^{-\frac{1}{2}}$. We prescribe initial data $(\phi(-2, \cdot), \partial_t\phi(-2, \cdot))$ on the time slice Σ_{-2} defined by $t = -2$. We use S_r to denote the sphere of radius r centered at the origin on Σ_{-2} and use B_2 to denote the ball of radius 2 with boundary S_2 . Therefore, the region enclosed by S_2 and $S_{2+\delta}$ (where δ is a small positive number) is foliated by the S_r 's for $2 \leq r \leq 2 + \delta$. The following picture may help to illustrate the process.



For each leaf S_r in the foliation, there is a unique incoming characteristic hypersurface, which will be defined more precisely in the next section, emanated from S_r . In the picture, we use a blue surface to denote it. The incoming characteristic hypersurfaces emanated from S_2 and $S_{2+\delta}$ are drawn in black and red respectively.

- (1) Data inside B_2 . We take trivial initial data $\phi(-2, x) \equiv 0$ and $\partial_t\phi(-2, x) \equiv 0$ inside B_2 .

In view of the Huygens' principle, in the backward solid light cone with bottom B_2 (colored in black in the picture), the solution ϕ is identically zero. In particular, for the incoming characteristic hypersurface,

which is the cone in black in the picture emanated from the leaf S_2 , since $c = (1 + 3G''(0)(\partial_t\phi)^2)^{-\frac{1}{2}}$, the incoming speed of this hypersurface, which is noted as c_1 , can be computed as

$$c_1 = 1.$$

(2) Data on the annulus region between S_2 and $S_{2+\delta}$. This is the region between the black circle and the red circle in the picture. We require that the size of $\partial_t\phi$ is approximately $\delta^{\frac{1}{2}}$ on the outermost circle (which is red in the picture) $S_{2+\delta}$, i.e. $|\partial_t\phi| \sim \delta^{\frac{1}{2}}$ at $S_{2+\delta}$.

By Taylor expansion, we can compute the speed c_2 of the outermost incoming characteristics hypersurface emanated from $S_{2+\delta}$ (which is red in the picture) as follows

$$c_2 = 1 - \frac{3}{2}G''(0)\delta + O(\delta^2).$$

We are now in a situation that resembles the Burgers' picture. The initial distance between the inner most characteristic hypersurface (which is black in the picture) and the outer most characteristic hypersurface (which is red in the picture) is δ . Both hypersurfaces travel towards the center. The difference of the speeds of two characteristic hypersurfaces is $c_1 - c_2 \sim \delta$. We also expect the "faster" (outer) characteristic hypersurface catching up the "slower" (inner) one. This catching up process needs approximately $\frac{\text{distance}}{\text{speed}} = \frac{\delta}{c_1 - c_2} \sim 1$ amount of time. We also regard the collision of characteristic hypersurfaces as shock formation, we hope that shocks form around $t = -1$.

We would like to point out a serious gap in the above heuristic argument. There is one assumption which seems to be very unreasonable: by the choice of the data, we can make sure that the speed c_2 of the outer most incoming characteristic hypersurfaces is of size $1 - \frac{3}{2}G''(0)\delta + O(\delta^2)$, but there is no clear reason that we should believe the speed c_2 remaining the same later on. Therefore, the difference of the two speeds $c_1 - c_2$ may vary a lot so that the outer most characteristic hypersurface never catch up with the inner one.

The whole point of the paper is to identify a set of initial data so that the profile of the data propagates, i.e. the profile remains almost unchanged. In particular, we can prove that the speeds of the characteristic hypersurfaces remain almost unchanged for later time. Another way to understand this is through energy estimates: we can find a specific set of data so that we can obtain a priori energy estimates. Once we showed that the energy (and its higher order analogue) is almost conserved, we can use Sobolev inequality to show that $\partial_t\phi$ is almost conserved pointwisely along the generators of the incoming characteristic hypersurfaces. According to the formula of c , this also implies the speeds are almost conserved.

Finally, we point out that, as in the Burgers' equation case, instead of showing that characteristic hypersurfaces meet, we show that the inverse density μ of the foliation by the characteristic hypersurfaces becomes 0, i.e. the foliation turns to being infinitely dense. Similarly, this can be done by showing that $\underline{L}\mu(t, x)$ is almost a constant along each generating geodesic of the characteristic hypersurfaces.

1.2. The main result. With motivations from the previous subsection, we are ready to state the main result of the paper. Let t be the time function in Minkowski spacetime. We use Σ_t to denote the level sets of t and it is a copy of \mathbb{R}^3 for each t . We fix $r_0 = 2$ in this paper.¹ We also use $\Sigma_{-r_0}^\delta$ to denote the following δ -thin annulus:

$$\Sigma_{-r_0}^\delta := \{x \in \Sigma_{-r_0} \mid r_0 \leq r(x) \leq r_0 + \delta\}, \quad (1.2)$$

where δ is any given small positive constant.

We recall that the wave speed c is defined as $c = (1 + 3G''(0)(\partial_t\phi)^2)^{-\frac{1}{2}}$. Let $\underline{L} = \partial_t - c\partial_r$ and $L = \partial_t + c\partial_r$. We first introduce a pair of functions $(\phi_1(s, \theta), \phi_2(s, \theta)) \in C^\infty((0, 1] \times \mathbb{S}^2)$ and we will call it the *seed data*.

¹In a future work, we will consider a more general case for which the data is prescribed at past infinity. Therefore, we have to let r_0 go to ∞ and the dependence of the estimates (of the current work) on r_0 will be crucial.

The seed data $(\phi_1(s, \theta), \phi_2(s, \theta))$ can be freely prescribed and once it is given once forever. In particular, the choice of the seed data is independent of the small parameter δ .

Lemma 1.1. *Given seed data (ϕ_1, ϕ_2) , there exists a $\delta' > 0$ depending only on the seed data, for all $\delta < \delta'$, we can construct another function $\phi_0 \in C^\infty((0, 1] \times \mathbb{S}^2)$ satisfying the following two properties:*

- (1) *For all $k \in \mathbb{Z}_{\geq 0}$, the C^k -norm of ϕ_0 are bounded by a function in the C^k -norms of ϕ_1 and ϕ_2 ;*
- (2) *If we pose initial data for (\star) on Σ_{-2} in the following way:*

For all $x \in \Sigma_{-2}$ with $r(x) \leq 2$, we require $(\phi(-2, x), \partial_t \phi(-2, x)) = (0, 0)$; For $2 \leq r(x) \leq 2 + \delta$, we require that

$$\phi(-2, x) = \delta^{3/2} \phi_0 \left(\frac{r-2}{\delta}, \theta \right), \quad (\partial_t \phi)(-2, x) = \delta^{1/2} \phi_1 \left(\frac{r-2}{\delta}, \theta \right).$$

Then we have

$$\|\underline{L}\phi\|_{L^\infty(\Sigma_{-r_0}^\delta)} \lesssim \delta^{3/2}, \quad \|\underline{L}^2\phi\|_{L^\infty(\Sigma_{-r_0}^\delta)} \lesssim \delta^{3/2}. \quad (1.3)$$

We remark that the condition (1.3) has a clear physical meaning: since \underline{L} are incoming directions, the waves are initially set to be incoming and the outgoing radiation is very little (controlled by δ).

Definition 1.2. *The Cauchy initial data of (\star) constructed in the lemma (satisfying the two properties) are called no-outgoing-radiation short pulse data*

Before we state the main theorem, we prove the lemma hence show the existence of no-outgoing-radiation short pulse data.

Proof. We recall that $c = (1 + 3G''(0)(\partial_t \phi)^2)^{-\frac{1}{2}}$, $\underline{L} = \partial_t - c\partial_r$ and $L = \partial_t + c\partial_r$.

We first take an arbitrary choice of ϕ_1 and fix this function. Therefore, $\partial_t \phi$ is given by the formula $(\partial_t \phi)(-2, x) = \delta^{1/2} \phi_1(\frac{r-2}{\delta}, \theta)$. In particular, all the spatial derivatives of $\partial_t \phi$ and c (determined completely by $\partial_t \phi$) are prescribed on Σ_{-2}^δ .

Therefore, by definition, we have

$$\underline{L}^2 \phi = \partial_t^2 \phi - \partial_t c \partial_r \phi + c \partial_r c \partial_r \phi + c^2 \partial_r^2 \phi - 2c \partial_r (\partial_t \phi).$$

According to the definition of c , we have $\partial_t c = -3G''(0)c^3 \partial_t \phi \partial_t^2 \phi$. Thus, we have

$$\underline{L}^2 \phi = (1 - 3G''(0)c^3 \partial_t \phi \partial_r \phi) \partial_t^2 \phi + c \partial_r c \partial_r \phi + c^2 \partial_r^2 \phi - 2c \partial_r (\partial_t \phi).$$

By virtue of the main equation, we have $\partial_t^2 \phi = c^2 (\partial_r^2 \phi + \frac{2}{r} \partial_r \phi + \frac{1}{r^2} \Delta_{\mathbb{S}^2} \phi)$, where $\Delta_{\mathbb{S}^2}$ is the Laplace operator on \mathbb{S}^2 . Therefore, we obtain

$$\begin{aligned} \underline{L}^2 \phi &= (2 - 3G''(0)c^3 \partial_t \phi \partial_r \phi) (c^2 \partial_r^2 \phi) + (1 - 3G''(0)c^3 \partial_t \phi \partial_r \phi) \left(\frac{2c^2}{r} \partial_r \phi + \frac{c^2}{r^2} \Delta_{\mathbb{S}^2} \phi \right) \\ &\quad + c \partial_r c \partial_r \phi - 2c \partial_r (\partial_t \phi). \end{aligned}$$

We now use the fact that $\phi(-2, x) = \delta^{\frac{3}{2}} \phi_0(\frac{r-2}{\delta}, \theta)$, where ϕ_0 will be determined later on. Thus, $\underline{L}^2 \phi$ can be computed as

$$\begin{aligned} \underline{L}^2 \phi &= (2 - 3G''(0)c^3 \phi_1 \partial_s \phi_0 \delta) (\delta^{-\frac{1}{2}} c^2 \partial_s^2 \phi_0) + (1 - 3G''(0)c^3 \phi_1 \partial_s \phi_0 \delta) \left(\frac{2c^2}{r} \partial_s \phi_0 \delta^{\frac{1}{2}} + \frac{c^2}{r^2} \Delta_{\mathbb{S}^2} \phi_0 \delta^{\frac{3}{2}} \right) \\ &\quad + c \partial_r c \partial_s \phi_0 \cdot \delta^{\frac{1}{2}} - 2c \partial_s \phi_1 \delta^{-\frac{1}{2}}. \end{aligned} \quad (1.4)$$

We claim that we can choose ϕ_0 , which may depend on the choice of ϕ_1 but is independent of δ , in such a way that $|\underline{L}^2 \phi| \lesssim \delta^{\frac{3}{2}}$.

To see this, we first observe that since ϕ_1 is given and δ is small, we have $|c| + |\partial_r c| \lesssim 1$. Indeed, $\partial_r c = -3c^3 G''(0) \phi_1 \partial_s \phi_1$ so the bound on $\partial_r c$ is clear. We make the following ansatz for ϕ_0 :

$$|\partial_s \phi_0| + |\partial_\theta^2 \phi_0| \leq C, \quad (1.5)$$

where the constant C may only depend on ϕ_1 but not on δ .

By the ansatz (1.5) and by looking at the expansions in $\delta^{\frac{1}{2}}$, one can ignore all the terms equal to or higher than $\delta^{\frac{3}{2}}$. Therefore, to show $|\underline{L}^2 \phi| \lesssim \delta^{\frac{3}{2}}$, it suffices to consider

$$(2 - 3G''(0)c^3 \phi_1 \partial_s \phi_0 \delta) (\delta^{-\frac{1}{2}} c^2 \partial_s^2 \phi_0) + \frac{2c^2}{r} \partial_s \phi_0 \delta^{\frac{1}{2}} + c \partial_r c \partial_s \phi_0 \cdot \delta^{\frac{1}{2}} - 2c \partial_s \phi_1 \delta^{-\frac{1}{2}} = O(\delta^{\frac{3}{2}}),$$

or equivalently

$$(2 - 3G''(0)c^3 \phi_1 \partial_s \phi_0 \delta) (c^2 \partial_s^2 \phi_0) + \frac{2c^2}{r} \partial_s \phi_0 \delta + c \partial_r c \partial_s \phi_0 \delta - 2c \partial_s \phi_1 = O(\delta^2).$$

Since $(2 - 3G''(0)c^3 \phi_1 \partial_s \phi_0 \delta)^{-1} = \frac{1}{2} + \frac{3}{4}G''(0)c^3 \phi_1 \partial_s \phi_0 \delta + O(\delta^2)$, by multiplying both sides of the above identity by $(2 - 3G''(0)c^3 \phi_1 \partial_s \phi_0 \delta)^{-1}$, it suffices to consider

$$c^2 \partial_s^2 \phi_0 + \frac{c^2}{r} \partial_s \phi_0 \delta + \frac{1}{2} c \partial_r c \partial_s \phi_0 \delta - \left(1 + \frac{3}{2}G''(0)c^3 \phi_1 \partial_s \phi_0 \delta\right) c \partial_s \phi_1 = O(\delta^2).$$

Since $c \sim 1$, we finally have

$$\partial_s^2 \phi_0 + \left(\frac{\delta}{r} + \frac{\delta}{2c} \partial_r c - \frac{3\delta}{2} G''(0)c^2 \phi_1 \partial_s \phi_1\right) \partial_s \phi_0 - c^{-1} \partial_s \phi_1 = O(\delta^2).$$

To solve for ϕ_0 , for $s \in [0, 1)$, we consider the following family of (parametrized by a compact set of parameters $\theta \in \mathbb{S}^2$ and the parameter δ) linear ordinary differential equation:

$$\begin{aligned} \partial_s^2 \phi_0 + \left(\frac{\delta}{r} + \frac{\delta}{2c} \partial_r c - \frac{3\delta}{2} G''(0)c^2 \phi_1 \partial_s \phi_1\right) \partial_s \phi_0 - c^{-1} \partial_s \phi_1 &= \delta^2 \phi_2, \\ \phi_0(0, \theta) &= 0, \quad \partial_s \phi_0(0, \theta) = 0. \end{aligned}$$

Since the C^k -norms of the solution depends smoothly on the coefficients and the parameter θ, δ , all C^k -norms of ϕ_0 are of order $O(1)$ and indeed are determined by the solution of

$$\begin{aligned} \partial_s^2 \phi_0 - c^{-1} \partial_s \phi_1 &= 0, \\ \phi_0(0, \theta) &= 0, \quad \partial_s \phi_0(0, \theta) = 0. \end{aligned}$$

In particular, this shows that the ansatz (1.5) holds if we choose C appropriately large in (1.5) and δ sufficiently small. Therefore the above construction shows that

$$|\underline{L}^2 \phi| \lesssim \delta^{\frac{3}{2}}.$$

We claim that, by the above choice of initial data, on Σ_{-2}^δ , we automatically have

$$|\underline{L} \phi| \lesssim \delta^{\frac{3}{2}}.$$

Indeed, by replacing $\partial_t = \underline{L} + c \partial_r$ in the main equation, we obtain

$$\partial_r \underline{L} \phi = \frac{1}{2c} \left(-\underline{L}^2 \phi - Lc \partial_r \phi + \frac{2c^2}{r} \partial_r \phi + \frac{c^2}{r^2} \Delta_{\mathbb{S}^2} \phi \right).$$

By the construction of the data, it is obvious that all the terms on the right hand side are of size $O(\delta^{\frac{1}{2}})$. By integrating from 2 to r with $r \in [2, 2 + \delta)$ and $\underline{L} \phi(2, \theta) = 0$, we have

$$|\underline{L} \phi(r, \theta)| \leq \delta \cdot O(\delta^{\frac{1}{2}}) \lesssim \delta^{\frac{3}{2}}.$$

□

The main theorem of the paper is as follows:

Main Theorem. *For a given constant $G''(0) \neq 0$, we consider*

$$-(1 + 3G''(0)(\partial_t \phi)^2)\partial_t^2 \phi + \Delta \phi = 0.$$

Let (ϕ_1, ϕ_2) be a pair of seed data and the initial data for the equation is taken to be the no-outgoing-radiation initial data.

If the following condition on ϕ_1 holds for at least one $(r, \theta) \in (0, 1] \times \mathbb{S}^2$:

$$G''(0) \cdot \partial_r \phi_1(r, \theta) \cdot \phi_1(r, \theta) \leq -\frac{1}{6}, \quad (1.6)$$

then there exists a constant δ_0 which depends only on the seed data (ϕ_1, ϕ_2) , so that for all $\delta < \delta_0$, shocks form for the corresponding solution ϕ before $t = -1$, i.e. ϕ will no longer be smooth.

Remark 1.3. *The choice of ϕ_1 in the proof of Lemma 1.1 is arbitrary. In particular, this is **consistent** with the condition (1.6) since ϕ_1 can be freely prescribed.*

Remark 1.4. (1) *We do not assume spherical symmetry on the initial data. Therefore, the theorem is in nature a higher dimensional result.*

(2) *The proof can be applied to a large family of equations derived through action principles. We will discuss this point when we consider the Lagrangian formulation of (\star) .*

(3) *The condition (1.6) is only needed to create shocks. It is not necessary at all for the a priori energy estimates.*

Remark 1.5. *The smoothness of ϕ breaks down in the following sense:*

- 1) *The solution and its first derivative, i.e. ϕ and $\partial \phi$, are always bounded. Moreover, $|\partial_t \phi| \lesssim \delta^{\frac{1}{2}}$, therefore (\star) is always of wave type.*
- 2) *The second derivative of the solution blows up. In fact, when one approaches the shocks, $\nabla \partial_t \phi$ blows up. See Remark 4.7 for the proof.*

1.3. Lagrangian formulation of the main equation and its relation to nonlinear electromagnetic waves. We briefly discuss the derivation of the main equation (\star) . The linear wave equation in Minkowski spacetime $(\mathbb{R}^{3+1}, m_{\mu\nu})$ can be derived by a variational principle: we take the Lagrangian density $L(\phi)$ to be $\frac{1}{2}(-(\partial_t \phi)^2 + |\nabla_x \phi|^2)$ and take the action functional $\mathcal{L}(\phi)$ to be $\int_{\mathbb{R}^{3+1}} L(\phi) d\mu_m$ where $d\mu_m$ denotes the volume form of the standard Minkowski metric $m_{\mu\nu}$. The corresponding Euler-Lagrange equation is exactly the linear wave equation $-\partial_t^2 \phi + \Delta \phi = 0$. We observe that the quadratic nature of the Lagrangian density result in the *linearity* of the equation. This simple observation allows one to derive plenty of *nonlinear* wave equations by changing the quadratic nature of the Lagrangian density. In particular, we will change the quadratic term in $\partial_t \phi$ to a quartic term, this will lead to a *quasi-linear* wave equation.

In fact, we consider a perturbation of the Lagrangian density of linear waves:

$$L(\phi) = -\frac{1}{2}G((\partial_t \phi)^2) + \frac{1}{2}|\nabla \phi|^2, \quad (1.7)$$

where $G = G(\rho)$ is a smooth function defined on \mathbb{R} and $\rho = |\partial_t \phi|^2$. The corresponding Euler-Lagrange equation is

$$-\partial_t(G'(\rho)\partial_t \phi) + \Delta \phi = 0.$$

The function $G(\rho)$ as a perturbation of $G_0(\rho) = \rho$ and therefore we can think of the above equation as a perturbation of the linear wave equation. For instance, we can work with a real analytic function $G(\rho)$ with $G(0) = 0$ and $G'(0) = 1$. In particular, we can perturb $G(\rho) = \rho$ in the simplest possible way by adding a quadratic function so that $G(\rho) = \rho + \frac{1}{2}G''(0)\rho^2$. In this situation, we obtain precisely the

main equation (\star) . It is in this sense that (\star) can be regarded as the simplest quasi-linear wave equation derived from action principles.

The main equation (\star) is also closely tight to electromagnetic waves in a nonlinear dielectric. The Maxwell equations in a homogeneous insulator derived from a Lagrangian L which is a function of the electric field E and the magnetic field B . The corresponding displacements D and H are defined through L by $D = -\frac{\partial L}{\partial E}$ and $H = \frac{\partial L}{\partial B}$ respectively. In the case of an isotropic dielectric, L is of the form

$$L = -\frac{1}{2}G(|E|^2) + \frac{1}{2}|B|^2, \quad (1.8)$$

hence $H = B$. The fields E and B are derived from the scalar potential ϕ and the vector potential A according to $E = -\nabla\phi - \partial_t A$ and $B = \nabla \times A$ respectively. This is equivalent to the first pair of Maxwell equations:

$$\nabla \times E + \partial_t B = 0, \quad \nabla \cdot B = 0.$$

The potentials are determined only up to a gauge transformation $\phi \mapsto \phi - \partial_t f$ and $A \mapsto A + df$, where f is an arbitrary smooth function. The second pair of Maxwell equations

$$\nabla \cdot D = 0, \quad \nabla \times H - \partial_t D = 0$$

are the Euler-Lagrange equations, the first resulting from the variation of ϕ and the second resulting from the variation of A . Fixing the gauge by setting $\phi = 0$, we obtain a simplified model if we neglect the vector character of A replacing it by a scalar function ϕ . Then the above equations for the fields in terms of the potentials simplify to $E = -\partial_t \phi$ and $B = \nabla \phi$. The Lagrangian (1.8) becomes

$$L = -\frac{1}{2}G((\partial_t \phi)^2) + \frac{1}{2}|\nabla \phi|^2$$

which is exactly (1.7). Therefore, the main equation (\star) provides a good approximation for shock formation in a natural physical model: the shock formations for nonlinear electromagnetic waves.

1.4. Main features of the proof. We now briefly sketch four main ingredients of the proof.

(1) The short pulse method. By rewriting (\star) in the semilinear form $\square_m \phi = -3G''(0)(\partial_t \phi)^2 \partial_t^2 \phi$, we notice that the nonlinearity is cubic. Therefore the result in [12] implies that small smooth initial data lead to global smooth solutions since the classical null condition is satisfied. We are then forced to consider large initial data. According to the choice of data in the Main Theorem, they are supported in the annulus of width δ and with amplitude $\delta^{\frac{1}{2}}$ which looks like a pulse (the short pulse data). The energy associated to the data is of size 1. On the technical level, although the short pulse data is no longer small, we still have a small parameter δ coming into play. Therefore, most of the techniques for small data problems can also be applied here.

(2) A Lorentzian geometry defined by the solutions. Since the Lagrangian, therefore (\star) itself, is invariant under the time translation and the isometries of \mathbb{R}^3 , we can linearize the equation via the infinitesimal generators of those actions. The most important feature about the linearized equations is, they are not just linear, they are linear *wave* equations with respect to a special Lorentzian metric defined by the solution. This reflects the Lagrangian nature of the equation (\star) : the metric comes from the second derivative of the Lagrangian. In particular, the incoming null hypersurfaces with respect to this metric correspond to the characteristic hypersurfaces of the solution. Recall that the shock formation is the study of the collapsing of the characteristic hypersurfaces, hence the differential geometry of the metric dictates shock formation.

Moreover, we study the energy estimates for the linearized linear wave equations. The energy estimates on one hand depend heavily on the underlying geometry, e.g. the curvature, the fundamental forms of null foliations, the isoperimetric inequalities, etc.; on the other hand, the energy estimates also control the underlying geometry. Therefore, the study of the linearized equations are more or less equivalent to the study of the underlying geometry. This leads to a natural bootstrap argument.

(3) **Coercivity of angular energy near shocks.** We use the vector field method to study the energy estimates for the linearized equations. Since we expect shock waves, the function μ , i.e., the inverse density of the characteristic hypersurfaces, may turn to 0. This will pose a fundamental difficulty for energy estimates (even for linear wave equations!). Roughly speaking, for a free wave ψ , for all possible multiplier vectorfields, in the associated energy or flux integrals, the components for the rotational directions all look like $\int \mu |\nabla \psi|^2$. But in the error integrals, some $\nabla \psi$ components show up *without* a μ factor. In view of the fact that $\mu \rightarrow 0$ in the shock region, the above disparity in μ shows that one can not control the error integrals by the energy or flux terms.

This difficulty is of course tied to the formation of shocks. The remarkable thing is, it is also resolved by the formation of shocks. The idea is as follows: initially, the $\mu \sim 1$. If in the future, no shock forms, then the disparity of μ simply result in an universal constant in the estimates since μ will be bounded below and above. If shock forms eventually, then along the incoming direction μ decreases, i.e. $\underline{L}\mu < 0$ where \underline{L} is the generator of the incoming null geodesics. Although the error integrals contain many terms without factor μ for $\nabla \psi$, there is one term has a very special form: it looks like $\iint \underline{L}\mu |\nabla \psi|^2$. The sign of $\underline{L}\mu$ in the shock region shows a miraculous coercivity of the energy estimates. This term is just enough to control all the $\nabla \psi$ terms appearing in the error terms. This is the major difference between the usual energy estimates and the case where shocks form. The use of the sign $\underline{L}\mu$ is the key to the entire argument in the current work.

(4) **The descent scheme.** The energy estimates on the top order terms may suffer a loss of a factor in μ and this can be dangerous in the shock region. Indeed, some error integral looks like $\int_{-r_0}^t \mu^{-1} \frac{\partial \mu}{\partial t} E(\tau) d\tau$ where $E(\tau)$ for the energy (it appears also on the lefthand side of the energy identity). If s^* is the time where shock forms, we can show that μ behaves like $|t - s^*|$ near shocks. Therefore, the presence of μ^{-1} cause a log loss in time. The descent scheme is designed to retrieve the loss. The idea is, rather than proving the top order terms are bounded in energy, we prove that the energy may blow up with a specific rate in μ to some negative power. To illustrate the idea, we do the following formal computations by assuming $\tilde{E}(t) = \sup_{\tau \leq t} \mu^a E(\tau)$ is bounded for a large positive number a . The energy identity, which looks like $E(t) + \dots \lesssim \int_{-r_0}^t \mu^{-1} \frac{\partial \mu}{\partial t} E(\tau) d\tau + \dots$, can be rewritten as

$$E(t) + \dots \lesssim \left(\int_{-r_0}^t \mu^{-(a+1)} \frac{\partial \mu}{\partial t} d\tau \right) \tilde{E}(t) + \dots$$

Since $\mu \sim |t - s^*|$, the term in the parenthesis gives a factor $\frac{1}{a}$ which is small (a is large!). Therefore, the righthand side can be absorbed by the left hand side.

2. THE OPTICAL GEOMETRY

2.1. Optical metrics and linearized equations. We observe that main equation (\star) is invariant under the following symmetries: space translations, rotations and the time translation. Indeed, the Lagrangian $L(\phi)$ is invariant under these symmetries, hence the Euler-Lagrange equation must be invariant too. We use A to denote any possible choice from $\{\partial_t, \partial_i, \Omega_{ij} = x^i \partial_j - x^j \partial_i\}$ where $i, j = 1, 2, 3$ and $i < j$. These vectorfields correspond to the infinitesimal generators of the symmetries of (\star) .

We linearize (\star) according to A by the following procedure: We apply the symmetry generated by A to a solution ϕ of (\star) to obtain a family of solutions $\{\phi_\tau : \tau \in \mathbb{R} | \phi_0 = \phi\}$. Therefore, $-\frac{1}{c(\phi_\tau)^2} \partial_t^2 \phi_\tau + \Delta \phi_\tau = 0$ for $\tau \in \mathbb{R}$. We then differentiate in τ and evaluate at $\tau = 0$. We define the so called *variations* ψ as

$$\psi := A\phi = \frac{d\phi_\tau}{d\tau} \Big|_{\tau=0} \quad (2.1)$$

By regarding ϕ as a fixed function, this procedure produce a linear equation for ψ . We call it *the linearized equation of (\star) for the solution ϕ with respect to the symmetry A* .

In the tangent space at each point in \mathbb{R}^{3+1} where the solution ϕ is defined, we introduce a Lorentzian metric $g_{\mu\nu}$ as follows

$$g = -c^2 dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3, \quad (2.2)$$

with (t, x^1, x^2, x^3) being the standard rectangular coordinates in Minkowski spacetime. Since c depends on the solution ϕ , $g_{\mu\nu}$ also depends on the solution ϕ . We also introduce a conformal metric $\tilde{g}_{\mu\nu}$ with the conformal factor $\Omega = \frac{1}{c}$

$$\tilde{g}_{\mu\nu} = \Omega \cdot g_{\mu\nu} = \frac{1}{c} g_{\mu\nu}. \quad (2.3)$$

We refer $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ as the *optical metric* and the *conformal optical metric* respectively.

Lemma 2.1. *The linearized equation of (\star) for a solution ϕ with respect to A can be written as*

$$\square_{\tilde{g}} \psi = 0, \quad (2.4)$$

where $\square_{\tilde{g}}$ is the wave operator with respect to \tilde{g} and $\psi = A\phi$.

There are two ways to derive the linearized equations.

To derive (2.4), we can directly differentiate (\star) . We denote the Christoffel symbols of \tilde{g} in the Cartesian coordinates by $\tilde{\Gamma}_{\alpha\beta}^\gamma$. Let $\tilde{\Gamma}^\gamma = \tilde{g}^{\alpha\beta} \tilde{\Gamma}_{\alpha\beta}^\gamma$, then $\tilde{\Gamma}^0 = -2c^{-2} \partial_t c$ and the other $\tilde{\Gamma}^\gamma$'s vanish. Hence,

$$\square_{\tilde{g}} \psi = \tilde{g}^{\mu\nu} \partial_\mu \partial_\nu \psi - \tilde{\Gamma}^\gamma \partial_\gamma \psi = c \left[\underbrace{\left(-\frac{1}{c^2} \partial_t^2 \psi + \Delta \psi \right)}_{T_1} - \underbrace{\partial_\rho \left(\frac{1}{c^2} \right) \partial_t \rho \cdot \partial_t \psi}_{T_2} \right],$$

where $\rho = (\partial_t \phi)^2$. We use A to differentiate (\star_g) . If A hits the factor $-\frac{1}{c^2}$, it yields T_2 ; since A (the symmetries!) commutes with ∂_t and Δ , the other terms are precisely T_1 .

There is a more natural proof which is standard in Lagrangian field theory, e.g. see [4]. In fact, the linearized equation of (\star) is the Euler-Lagrange equation of the linearized Lagrangian density $\dot{L}(\psi) := \frac{1}{2} \frac{d^2}{d\tau^2} \big|_{\tau=0} L(\phi + \tau\psi)$. Since $G(\rho) = \rho + \frac{1}{2} G''(0) \rho^2$, we have

$$\dot{L}(\psi) = -\frac{1}{2} G'(0) (\partial_t \psi)^2 - \frac{3}{2} G''(0) (\partial_t \phi)^2 (\partial_t \psi)^2 + \frac{1}{2} |\nabla \psi|^2 = \frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi.$$

Therefore, if $D \subset \mathbb{R}^{3+1}$ is a domain in which the solution ϕ is defined, the action corresponding to \dot{L} is $\dot{\mathcal{L}}(\psi) = \frac{1}{2} \int_D g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi d\mu_m$. We emphasize that the volume form $d\mu_m$ is defined by the Minkowski metric $m_{\alpha\beta}$. In view of the definitions of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, the action $\dot{\mathcal{L}}(\psi)$ can be written as

$$\dot{\mathcal{L}}(\psi) = \frac{1}{2} \int_D \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi d\mu_{\tilde{g}}.$$

At this stage, it is clear that the linearized equation is the free wave equation with respect to $\square_{\tilde{g}}$.

2.2. Lorentzian geometry of the maximal development.

2.2.1. *The maximal development.* We define a function \underline{u} on Σ_{-2} as follows:

$$\underline{u} := r - 2. \quad (2.5)$$

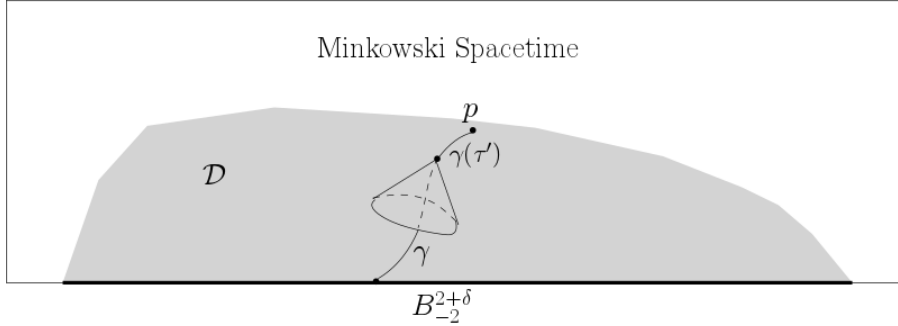
The level sets of \underline{u} in Σ_{-2} are denoted by $S_{-2, \underline{u}}$ and they are round spheres of radii $\underline{u} + 2$. The annular region Σ_{-2}^δ defined in (1.2) is foliated by $S_{-2, \underline{u}}$ as

$$\Sigma_{-2}^\delta := \bigcup_{\underline{u} \in [0, \delta]} S_{-2, \underline{u}}. \quad (2.6)$$

Given an initial data set $(\phi, \partial_t \phi)|_{t=-2}$ defined on $B_{-2}^{2+\delta} = \bigcup_{\underline{u} \in [-2, \delta]} S_{-2, \underline{u}}$ to the main equation (\star) (as we stated in the Main Theorem), we recall the notion of *the maximal development* or maximal solution with respect to the given data.

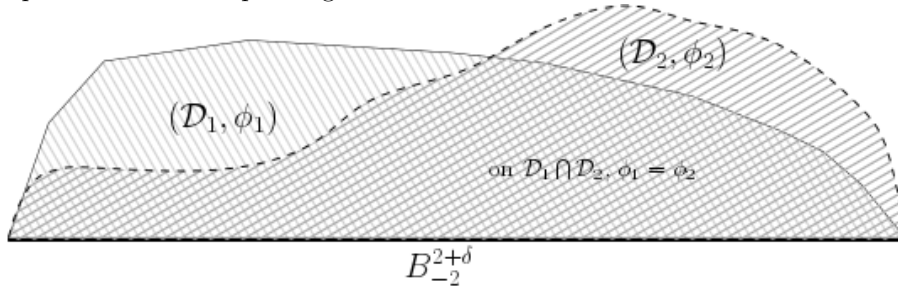
By virtue of the local existence theorem (to (\star) with smooth data), one can claim the existence of a *development* of the given initial data set, namely, the existence of

- a domain \mathcal{D} in Minkowski spacetime, whose past boundary is $B_{-2}^{2+\delta}$;
- a smooth solution ϕ to (\star) defined on \mathcal{D} with the given data on $B_{-2}^{2+\delta}$ with following property:
For any point $p \in \mathcal{D}$, if an inextendible curve $\gamma : [0, \tau) \rightarrow \mathcal{D}$ satisfies the property that
 - (1) $\gamma(0) = p$,
 - (2) For any $\tau' \in [0, \tau)$, the tangent vector $\gamma'(\tau')$ is past-pointed and causal (i.e., $g(\gamma'(\tau'), \gamma'(\tau')) \leq 0$) with respect to the optical metric $g_{\alpha\beta}$ at the point $\gamma(\tau')$,
 then the curve γ must terminate at a point of $B_{-2}^{2+\delta}$.



By the standard terminology of Lorentzian geometry, the above simply says that $B_{-2}^{2+\delta}$ is a Cauchy hypersurface of \mathcal{D} .

The local uniqueness theorem asserts that if (\mathcal{D}_1, ϕ_1) and (\mathcal{D}_2, ϕ_2) are two developments of the same initial data sets, then $\phi_1 = \phi_2$ in $\mathcal{D}_1 \cap \mathcal{D}_2$. Therefore the union of all developments of a given initial data set is itself a development. This is the so called *maximal development* and its corresponding domain is denoted by W^* . The corresponding solution is called the *maximal solution*. Sometimes we also identify the development as its corresponding domain when there is no confusion.



2.2.2. Geometric set-up. Given an initial data set, we consider a specific family of incoming null hypersurfaces (with respect to the optical metric g) on the maximal development W^* . Recall that \underline{u} is defined on Σ_{-2} as $r - 2$. For any $\underline{u} \in [0, \delta]$, we use $\underline{C}_{\underline{u}}$ to denote the incoming null hypersurface emanated from the sphere $S_{-2, \underline{u}}$. By definition, we have $\underline{C}_{\underline{u}} \subset W^*$ and $\underline{C}_{\underline{u}} \cap \Sigma_{-2} = S_{-2, \underline{u}}$.

We denote the subset of the maximal development of the given initial data foliated by $\underline{C}_{\underline{u}}$ with $\underline{u} \in [0, \delta]$ by W_δ , i.e.,

$$W_\delta = \bigcup_{\underline{u} \in [0, \delta]} \underline{C}_{\underline{u}}. \quad (2.7)$$

The diagram is a Penrose diagram with a vertical axis labeled $t = t^*$ at the top and $t = -2$ at the bottom. The horizontal axis is labeled $r = 0 \text{ or } u = -2$ on the left, $u = 0$ in the middle, and $u = \delta$ on the right. A shaded region W_δ is bounded by a null surface L (dashed line) and a timelike surface C_δ (solid line). A horizontal line Σ_t intersects L and C_δ . The region to the left of Σ_t is labeled $\phi \equiv 0$. The region to the right of Σ_t is labeled W_δ . A point on Σ_t is labeled $S_{t,u}$. A curve C_u is shown, with a segment labeled C_u^t . A dashed line L is also shown, with a segment labeled L . A vertical double-headed arrow on the right is labeled W_δ^* .

The dashed line denotes a incoming null hypersurface in the above picture. We extend the function \underline{u} to W_δ by requiring that the hypersurfaces $\underline{C}_{\underline{u}}$ are precisely the level sets of the function \underline{u} . Since $\underline{C}_{\underline{u}}$ is null with respect to $g_{\alpha\beta}$, the function \underline{u} is then a solution to the equation

$$(g^{-1})^{\alpha\beta} \partial_\alpha \underline{u} \partial_\beta \underline{u} = 0, \quad (2.8)$$

With respect to the affine parameter, the future-directed tangent vectorfield of a null geodesic on \underline{C}_u is given by

$$\hat{\underline{L}} := -(g^{-1})^{\alpha\beta} \partial_{\alpha} \underline{u} \partial_{\beta}. \quad (2.9)$$

$$\underline{L} = \mu \hat{L}, \quad \underline{L}t = 1, \quad (2.10)$$

The function μ can be computed as

$$\frac{1}{\mu} = -(g^{-1})^{\alpha\beta} \partial_\alpha \underline{u} \partial_\beta t.$$

Given $\underline{u} \leq \delta$, to consider the density of null-hypersurface-foliation on $\Sigma_t \cap W_\delta$, we define

$$\mu_m^u(t) = \min \left(\inf_{(\underline{u}', \theta) \in [0, u] \times \mathbb{S}^2} \mu(t, \underline{u}', \theta), 1 \right). \quad (2.11)$$

$$s_* = \sup \{t | t \geq -2 \text{ and } \mu_m^\delta(t) > 0\}.$$

From the PDE perspective, for the given initial data to (\star) (as constructed in Lemma 1.1), we also define

$t_* = \sup \{ \tau \mid \tau \geq -2 \text{ such that the smooth solution exists for all } (t, \underline{u}) \in [-2, \tau) \times [0, \delta] \text{ and } \theta \in \mathbb{S}^2 \}.$

Finally, we define

$$s^* = \min\{s_*, -1\}, \quad t^* = \min\{t_*, s^*\}. \quad (2.12)$$

We remark that we will exhibit data in such a way that the solution breaks down before $t = -1$. This is the reason we take -1 in the definition of s^* .

In the sequel, we will work in a further confined spacetime domain $W_\delta^* \subset W_\delta \subset W^*$ to prove a priori energy estimates. By definition, it consists of all the points in W_δ with time coordinate $t \leq t^*$, i.e.,

$$W_\delta^* = W_\delta \cap \left(\bigcup_{-2 \leq t \leq t^*} \Sigma_t \right).$$

In the previous picture, the region W_δ^* is the part of the grey region below the horizontal dash-dot line.

For the purpose of future use, we introduce more notations to describe various geometric objects.

For each $(t, \underline{u}) \in [-2, t^*) \times [0, \delta]$, we use $S_{t, \underline{u}}$ to denote the closed two dimensional surface

$$S_{t, \underline{u}} := \Sigma_t \cap \underline{C}_{\underline{u}}. \quad (2.13)$$

In particular, we have

$$W_\delta^* = \bigcup_{(t, \underline{u}) \in [-2, t^*) \times [0, \delta]} S_{t, \underline{u}}. \quad (2.14)$$

For each $(t, \underline{u}) \in [-2, t^*) \times [0, \delta]$, we define

$$\begin{aligned} \Sigma_t^{\underline{u}} &:= \{ (t, \underline{u}', \theta) \in \Sigma_t \mid 0 \leq \underline{u}' \leq \underline{u} \}, \\ \underline{C}_{\underline{u}}^t &:= \{ (\tau, \underline{u}, \theta) \in \underline{C}_{\underline{u}} \mid -2 \leq \tau \leq t \}, \\ W_{\underline{u}}^t &:= \bigcup_{(t', \underline{u}') \in [-2, t) \times [0, \underline{u}]} S_{t', \underline{u}'}. \end{aligned} \quad (2.15)$$

One can consult the previous picture to visualize those objects. In particular, $W_{\underline{u}}^t$ is the grey region bounded by $\Sigma_t^{\underline{u}}$ and $\underline{C}_{\underline{u}}^t$.

In what follows when working in $W_{\underline{u}}^t$, we usually omit the superscript \underline{u} to write $\mu_m^{\underline{u}}(t)$ as $\mu_m(t)$, whenever there is no confusion.

We define the vectorfield T in W_δ^* by the following three conditions:

- 1) T is tangential to Σ_t ;
- 2) T is orthogonal (with respect to g) to $S_{t, \underline{u}}$ for each $\underline{u} \in [0, \delta]$;
- 3) $T\underline{u} = 1$.

The letter T stands for “transversal” since the vectorfield is transversal to the foliation of null hypersurfaces $\underline{C}_{\underline{u}}$.

In particular, the point 1) implies

$$Tt = 0. \quad (2.16)$$

According to (2.8)-(2.10), we have

$$\underline{L}\underline{u} = 0, \quad \underline{L}t = 1. \quad (2.17)$$

In view of (2.10), (2.17), (2.16) and the fact $T\underline{u} = 1$, we see that the commutator

$$\Lambda := [\underline{L}, T] \quad (2.18)$$

is tangential to $S_{t, \underline{u}}$.

In view of (2.8)-(2.10) and the fact $T\underline{u} = 1$ we have

$$g(\underline{L}, T) = -\mu, \quad g(\underline{L}, \underline{L}) = 0. \quad (2.19)$$

Since T is spacelike with respect to g (indeed, Σ_t is spacelike and T is tangential to Σ_t), we denote

$$g(T, T) = \kappa^2, \quad \kappa > 0. \quad (2.20)$$

Lemma 2.2. *We have the following relations for \underline{L} , T , μ and κ :*

$$\mu = c\kappa, \quad \underline{L} = \partial_0 - c\kappa^{-1}T, \quad (2.21)$$

where ∂_0 is the standard time vectorfield in Minkowski spacetime.

Proof. The vectorfield ∂_0 is perpendicular to Σ_t and therefore is perpendicular to $S_{t, \underline{u}}$. Since \underline{L} and T are two linearly independent vectorfields perpendicular to $S_{t, \underline{u}}$ and $\underline{L}t = \partial_0 t = 1$, we have

$$\partial_0 = \underline{L} + fT$$

for some scalar function f . On the other hand, ∂_0 is perpendicular to Σ_t hence to T , we have

$$0 = g(\partial_0, T) = g(\underline{L}, T) + fg(T, T) = -\mu + f\kappa^2.$$

Therefore, $f = \frac{\mu}{\kappa^2} = \frac{c}{\kappa}$ and the second formula in (2.21) follows.

For the first formula, in view of the defining equation of the optical metric g , we have

$$-c^2 = g(\partial_0, \partial_0) = g(\underline{L} + fT, \underline{L} + fT) = -2f\mu + f^2\kappa^2.$$

Since $f = \frac{c}{\kappa}$, we can solve for μ to complete the proof. \square

Remark 2.3. *On the initial Cauchy surface Σ_{-2} , since $\underline{u} = r - 2$, we have $T = \partial_r$ and $\kappa = 1$. Therefore, by using the standard rectangular coordinates, we obtain that*

$$\underline{L} = \partial_t - c\partial_r.$$

This is coherent with the notations and computations in Lemma 1.1.

2.2.3. The optical coordinates. We construct a new coordinate system on W_δ^* . If shocks form, the new coordinate system is completely different from the rectangular coordinates. Indeed, we will show that they define two differentiable structures on W_δ^* when shocks form.

Given $\underline{u} \in [0, \delta]$, the generators of $\underline{C}_{\underline{u}}$ define a diffeomorphism between $S_{-2, \underline{u}}$ and $S_{t, \underline{u}}$ for each $t \in [-2, t^*]$. Since $S_{-2, \underline{u}}$ is diffeomorphic to the standard sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ in a natural way. We obtain a natural diffeomorphism between $S_{t, \underline{u}}$ and \mathbb{S}^2 . If local coordinates (θ^1, θ^2) are chosen on \mathbb{S}^2 , the diffeomorphism then induces local coordinates on $S_{t, \underline{u}}$ for every $(t, \underline{u}) \in [-2, t^*] \times [0, \delta]$. The local coordinates (θ^1, θ^2) , together with the functions (t, \underline{u}) define a complete system of local coordinates $(t, \underline{u}, \theta^1, \theta^2)$ for W_δ^* . This new coordinates are defined as *the optical coordinates*.

We now express for \underline{L} , T and the optical metric g in the optical coordinates.

First of all, the integral curves of \underline{L} are the lines with constant \underline{u} and θ . Since $\underline{L}t = 1$, therefore in optical coordinates we have

$$\underline{L} = \frac{\partial}{\partial t}. \quad (2.22)$$

Similarly, since $T\underline{u} = 1$ and T is tangential to Σ_t , we have

$$T = \frac{\partial}{\partial \underline{u}} - \Xi \quad (2.23)$$

with Ξ a vectorfield tangential to $S_{t,\underline{u}}$. Locally, we can express Ξ as

$$\Xi = \sum_{A=1,2} \Xi^A \frac{\partial}{\partial \theta^A}, \quad (2.24)$$

The metric g then can be written in the optical coordinates $(t, \underline{u}, \theta^1, \theta^2)$ as

$$g = -2\mu dt d\underline{u} + \kappa^2 d\underline{u}^2 + \not{g}_{AB} (d\theta^A + \Xi^A d\underline{u}) (d\theta^B + \Xi^B d\underline{u}) \quad (2.25)$$

with

$$\not{g}_{AB} = g \left(\frac{\partial}{\partial \theta^A}, \frac{\partial}{\partial \theta^B} \right), \quad 1 \leq A, B \leq 2. \quad (2.26)$$

To study the differentiable structure defined by the optical coordinates, we study the Jacobian Δ of the transformation from the optical coordinates $(t, \underline{u}, \theta^1, \theta^2)$ to the rectangular coordinates (x^0, x^1, x^2, x^3) .

First of all, since $x^0 = t$, we have

$$\frac{\partial x^0}{\partial t} = 1, \quad \frac{\partial x^0}{\partial \underline{u}} = \frac{\partial x^0}{\partial \theta^A} = 0.$$

Secondly, by (2.23), we can express $T = T^i \partial_i$ in the rectangular coordinates (x^1, x^2, x^3) as

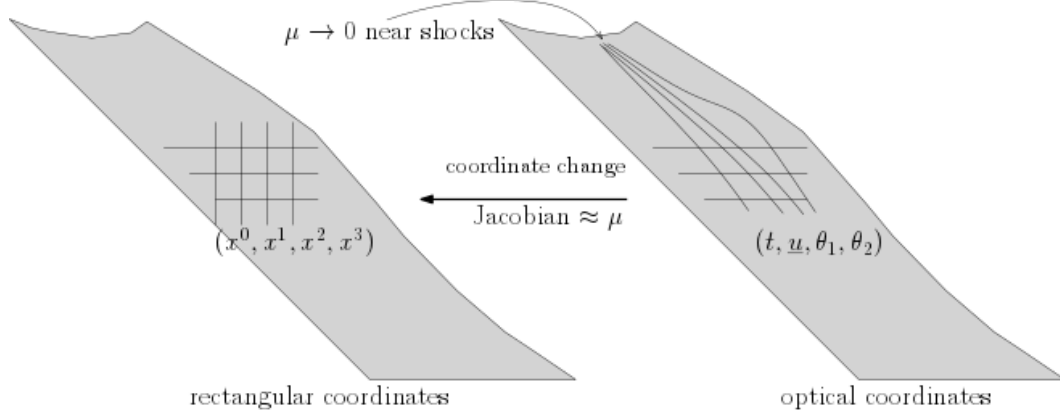
$$T^i = \frac{\partial x^i}{\partial \underline{u}} - \sum_{A=1,2} \Xi^A \frac{\partial x^i}{\partial \theta^A}$$

In view of the fact that T is orthogonal to $\frac{\partial}{\partial \theta^A}$ with respect to the Euclidean metric (which is the induced metric of g on Σ_t !), we have

$$\Delta = \det \begin{pmatrix} T^1 & T^2 & T^3 \\ \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^1} \\ \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^2} \end{pmatrix} = \|T\| \left\| \frac{\partial}{\partial \theta^1} \wedge \frac{\partial}{\partial \theta^2} \right\| = c^{-1} \mu \sqrt{\det \not{g}},$$

where $\|\cdot\|$ measures the magnitude of a vectorfield with respect to the Euclidean metric in \mathbb{R}^3 (defined by the rectangular coordinates (x^1, x^2, x^3)).

We end the discussion by an important remark. We can also read the conclusions from the following picture:



Remark 2.4 (Geometric meaning of μ). *In the sequel, we will show that the wave speed function c will be always approximately equal to 1 in W_δ^* . Since $\mu = c\kappa$, we may think of μ being κ in a efficient way.*

On the other hand, by the definition of T , in particular $T\underline{u} = 1$, we know that κ^{-1} is indeed the density of the foliation by the $\underline{C}_{\underline{u}}$'s. This is because $g(T, T) = \kappa^2$. Since the optical metric coincides with the Euclidean metric on each constant time slice Σ_t , by $\mu \sim \kappa$, we arrive at the following conclusion:

- μ^{-1} measures the foliation of the incoming null hypersurfaces $\underline{C}_{\underline{u}}$'s.

Therefore, by regarding shock formation as the collapsing (i.e. the density blows up) of the characteristics (\simeq the incoming null hypersurfaces), we may say that

- Shock formation is equivalent to $\mu \rightarrow 0$.

By virtue of the formula $\Delta = c^{-1}\mu\sqrt{\det g}$, it is clear (the volume element $\sqrt{\det g}$ will be controlled in the sequel) that if shock forms then the coordinate transformation between the optical coordinates and the rectangular coordinates will fail to be a diffeomorphism. Therefore, we can also say that

- Shock formation is equivalent to the fact that the optical coordinates on the maximal development defines a different differentiable structure (compared to the usual differentiable structure induced from the Minkowski spacetime).

2.3. Connection, curvature and structure equations. We use ∇ to denote the Levi-Civita connection of g and use X_A to denote $\frac{\partial}{\partial \theta^A}$. The 2nd fundamental form of the embedding $S_{t, \underline{u}} \hookrightarrow \underline{C}_{\underline{u}}$ is

$$\underline{\chi}_{AB} = g(\nabla_{X_A} \underline{L}, X_B). \quad (2.27)$$

The trace/traceless part is defined by $\text{tr} \underline{\chi} = \text{tr}_g \underline{\chi} = g^{AB} \underline{\chi}_{AB}$ and $\hat{\underline{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2} \text{tr} \underline{\chi} g_{AB}$. Let $\hat{T} = c\mu^{-1}T$. Then, $g(\hat{T}, \hat{T}) = 1$. The 2nd fundamental form of $S_{t, \underline{u}} \hookrightarrow \Sigma_t$ is

$$\theta_{AB} = g(\nabla_{X_A} \hat{T}, X_B) \quad (2.28)$$

By virtue of (2.2), we have $\underline{\chi}_{AB} = -c\theta_{AB}$. Thanks to Gauss' Theorema Egregium, the Gauss curvature K of $S_{t, \underline{u}}$ is

$$K = \frac{1}{2}(\text{tr}_g \theta)^2 - \frac{1}{2}|\theta|_g^2 = \frac{1}{2}c^{-2}((\text{tr}_g \underline{\chi})^2 - |\underline{\chi}|_g^2). \quad (2.29)$$

We introduce an outgoing null vectorfield

$$L = c^{-2}\mu\underline{L} + 2T \quad (2.30)$$

so that $g(L, \underline{L}) = -2\mu$. The corresponding 2nd fundamental form is $\chi_{AB} = g(\nabla_{X_A} L, X_B)$. Similarly, we define $\text{tr} \chi = \text{tr}_g \chi = g^{AB} \chi_{AB}$ and $\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} \text{tr} \chi g_{AB}$.

The torsion one forms $\underline{\eta}_A$ and $\underline{\zeta}_A$ are defined by $\underline{\zeta}_A = g(\nabla_{X_A} \underline{L}, T)$ and $\underline{\eta}_A = -g(\nabla_{X_A} T, \underline{L})$. They are related to the inverse density μ by $\underline{\eta}_A = \underline{\zeta}_A + X_A(\mu)$ and $\underline{\zeta}_A = -c^{-1}\mu X_A(c)$.

The covariant derivative ∇ is now expressed in the frame $(T, \underline{L}, X_1, X_2)$ by (∇ is the restriction of ∇ on $S_{t, \underline{u}}$)

$$\begin{aligned} \nabla_{\underline{L}} \underline{L} &= \mu^{-1}(\underline{L}\mu)\underline{L}, \quad \nabla_T \underline{L} = \underline{\eta}^A X_A - c^{-1}\underline{L}(c^{-1}\mu)\underline{L}, \\ \nabla_{X_A} \underline{L} &= -\mu^{-1}\underline{\zeta}_A \underline{L} + \underline{\chi}_A{}^B X_B, \quad \nabla_{\underline{L}} T = -\underline{\zeta}^A X_A - c^{-1}\underline{L}(c^{-1}\mu)\underline{L}, \\ \nabla_T T &= c^{-3}\mu(Tc + \underline{L}(c^{-1}\mu))\underline{L} + \left(c^{-1}(Tc + \underline{L}(c^{-1}\mu)) + T(\log(c^{-1}\mu))\right)T - c^{-1}\mu g^{AB} X_B(c^{-1}\mu)X_A, \\ \nabla_{X_A} T &= \mu^{-1}\underline{\eta}_A T + c^{-1}\mu\theta_{AB}g^{BC}X_C, \quad \nabla_{\underline{L}} X_A = \nabla_{X_A} \underline{L}, \quad \nabla_{X_A} X_B = \nabla_{X_A} X_B + \mu^{-1}\underline{\chi}_{AB} T. \end{aligned}$$

In terms of null frames $(L, \underline{L}, X_1, X_2)$, we have

$$\begin{aligned}\nabla_L \underline{L} &= -\underline{L}(c^{-2}\mu)\underline{L} + 2\eta^A X_A, \quad \nabla_{\underline{L}} L = -2\zeta^A X_A, \\ \nabla_L L &= (\mu^{-1}L\mu + \underline{L}(c^{-2}\mu))L - 2\mu X^A(c^{-2}\mu)X_A, \quad \nabla_{X_A} L = \mu^{-1}\underline{\eta}_A L + \chi_A{}^B X_B, \\ \nabla_{X_A} X_B &= \nabla_{X_A} X_B + \frac{1}{2}\mu^{-1}\underline{\chi}_{AB}L + \frac{1}{2}\mu^{-1}\chi_{AB}\underline{L}.\end{aligned}$$

In the Cartesian coordinates, the only non-vanishing curvature components are R_{0i0j} 's:

$$R_{0i0j} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_i \nabla_j \rho + \frac{1}{2} \frac{d^2(c^2)}{d\rho^2} \nabla_i \rho \nabla_j \rho - \frac{1}{4} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \nabla_i \rho \nabla_j \rho.$$

In the optical coordinates, the only nonzero curvature components are $\underline{\alpha}_{AB} = R(X_A, \underline{L}, X_B, \underline{L})$:

$$\underline{\alpha}_{AB} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_{X_A, X_B}^2 \rho - \frac{1}{2} \mu^{-1} \frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB} + \frac{1}{2} \left(\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \right) X_A(\rho) X_B(\rho).$$

We define

$$\underline{\alpha}'_{AB} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_{X_A, X_B}^2 \rho + \frac{1}{2} \left[\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \right] X_A(\rho) X_B(\rho).$$

Therefore, we write

$$\underline{\alpha}_{AB} = -\frac{1}{2} \mu^{-1} \frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB} + \underline{\alpha}'_{AB}. \quad (2.31)$$

Remark 2.5. As a convention, we say that the first term on the right hand side of (2.31) is singular in μ (since μ may go to zero). The second term $\underline{\alpha}'_{AB}$ is regular in μ .

Indeed, in the course of the proof, we will see that $\underline{\alpha}'_{AB}$ are bounded and $\underline{\alpha}_{AB}$ behaves exactly as μ^{-1} in amplitude. Therefore, in addition to two equivalent descriptions of the shock formation in Remark 2.4, we have another geometric interpretation:

- Shock formation is equivalent to the fact that curvature tensor of the optical metric g becomes unbounded.

Compared to the one dimensional picture of shock formations in conservation laws, e.g., for inviscid Burgers equation, this new description of shock formation is purely geometric in the following sense: it does not even depend on the choice of characteristic foliation (because the curvature tensor is tensorial!).

In the frame $(T, \underline{L}, \frac{\partial}{\partial \theta^A})$, the connection coefficients and the curvature components satisfies the following structure equations:

$$\underline{L}(\underline{\chi}_{AB}) = \mu^{-1}(\underline{L}\mu)\underline{\chi}_{AB} + \underline{\chi}_A{}^C \underline{\chi}_{BC} - \underline{\alpha}_{AB}, \quad (2.32)$$

$$d\text{iv} \underline{\chi} - d\text{tr} \underline{\chi} = -\mu^{-1}(\zeta \cdot \underline{\chi} - \zeta \text{tr} \underline{\chi}), \quad (2.33)$$

$$\text{L}^T \underline{\chi}_{AB} = (\nabla \widehat{\otimes} \eta)_{AB} + \mu^{-1}(\zeta \widehat{\otimes} \eta)_{AB} - c^{-1} \underline{L}(c^{-1}\mu)\underline{\chi}_{AB} + c^{-1} \mu(\theta \widehat{\otimes} \underline{\chi})_{AB}, \quad (2.34)$$

where $(\zeta \cdot \underline{\chi})_B = \not{g}^{AC} \zeta_A \underline{\chi}_{BC}$, $(\nabla \widehat{\otimes} \eta)_{AB} = \frac{1}{2}(\nabla_A \eta_B + \nabla_B \eta_A)$, $(\zeta \widehat{\otimes} \eta)_{AB} = \frac{1}{2}(\zeta_A \eta_B + \zeta_B \eta_A)$ and $(\theta \widehat{\otimes} \underline{\chi})_{AB} = \frac{1}{2}(\theta_{AC} \underline{\chi}_B^C + \theta_{BC} \underline{\chi}_A^C)$. By taking the trace of (2.32), we have

$$\underline{L} \text{tr} \underline{\chi} = \mu^{-1}(\underline{L}\mu) \text{tr} \underline{\chi} - |\underline{\chi}|_{\not{g}}^2 - \text{tr} \underline{\alpha}. \quad (2.35)$$

The inverse density function μ satisfies the following transport equation:

$$\underline{L}\mu = m + \mu e, \quad (2.36)$$

with $m = -\frac{1}{2} \frac{d(c^2)}{d\rho} T\rho$ and $e = \frac{1}{2c^2} \frac{d(c^2)}{d\rho} \underline{L}\rho$. With these notations, we have $\underline{\alpha}_{AB} = \mu^{-1} m \underline{\chi}_{AB} + \underline{\alpha}'_{AB}$.

Regarding the regularity in μ , we use (2.36) to replace $\underline{L}\mu$ in (2.32). This yields

$$\underline{L}(\underline{\chi}_{AB}) = e \underline{\chi}_{AB} + \underline{\chi}_A{}^C \underline{\chi}_{BC} - \underline{\alpha}'_{AB}. \quad (2.37)$$

Compared to the original (2.32), the new equation is *regular* μ in the sense that it has no μ^{-1} terms.

2.4. Rotation Vectorfields. Although $g|_{\Sigma_t}$ is flat, the foliation $S_{t,\underline{u}}$ is different from the standard spherical foliations. In the Cartesian coordinates on Σ_t , let $\Omega_1 = x^2\partial_3 - x^3\partial_2$, $\Omega_2 = x^3\partial_1 - x^1\partial_3$ and $\Omega_3 = x^1\partial_2 - x^2\partial_1$ be the standard rotations. Let Π be the orthogonal projection to $S_{t,\underline{u}}$ (embedded in Σ_t). The rotation vectorfields $R_i \in \Gamma(TS_{t,\underline{u}})$ ($i = 1, 2, 3$) are defined by

$$R_i = \Pi \Omega_i. \quad (2.38)$$

Let indices $i, j, k \in \{1, 2, 3\}$. We use the T^k , \underline{L}^k and X_A^k to denote the components for T , \underline{L} and X_A in the Cartesian frame $\{\partial_i\}$ on Σ_t (notice that \underline{L} has also a 0 component $L^0 = 1$). We use $\hat{T} = c\mu^{-1}T$ is the outward unit normal of $S_{t,\underline{u}}$ in Σ_t . We introduce some functions to measure the difference between the foliations $S_{t,\underline{u}}$ and the standard spherical foliations.

The functions λ_i 's measure the derivation from R_i to Ω_i :

$$\lambda_i \hat{T} = \Omega_i - R_i. \quad (2.39)$$

The functions y'^k 's measure the derivation from \hat{T} to the standard radial vectorfield $\frac{x^i}{r}\partial_i$:

$$y'^k = \hat{T}^k - \frac{x^k}{r}. \quad (2.40)$$

We also define (we will show that $|y^k - y'^k|$ is bounded by a negligible small number)

$$y^k = \hat{T}^k - \frac{x^k}{\underline{u} - t}. \quad (2.41)$$

The functions z^k 's measure the derivation of \underline{L} from $\partial_t - \partial_r$ in Minkowski spacetime:

$$z^k = \underline{L}^k + \frac{x^k}{\underline{u} - t} = -\frac{(c-1)x^k}{\underline{u} - t} - cy^k.$$

Finally, the rotation vectorfields can be expressed as

$$R_i = \Omega_i - \lambda_i \sum_{j=1}^3 \hat{T}^j \partial_j, \quad \lambda_i = \sum_{j,k,l=1}^3 \varepsilon_{ilk} x^l y^k, \quad (2.42)$$

where ε_{ijk} is the totally skew-symmetric symbol.

3. INITIAL DATA, BOOTSTRAP ASSUMPTIONS AND THE MAIN ESTIMATES

3.1. Preliminary estimates on initial data. In the Main Theorem, we take the so called short pulse datum for (\star) on $\Sigma_{-r_0}^\delta$. Recall that $\phi(-r_0, x) = \delta^{3/2}\phi_0(\frac{r-r_0}{\delta}, \theta)$ and $\partial_t\phi(-r_0, x) = \delta^{1/2}\phi_1(\frac{r-r_0}{\delta}, \theta)$, where $\phi_0, \phi_1 \in C_0^\infty((0, 1] \times \mathbb{S}^2)$. The condition (3) in the statement of the Main Theorem reads as

$$\|\underline{L}\phi\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta^{3/2}, \quad \|\underline{L}^2\phi\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta^{3/2}.$$

We now derive estimates for ϕ and its derivatives on Σ_{-2}^δ . These estimates also suggest the estimates, e.g. the bootstrap assumptions in next subsection, that one can expect later on.

For ϕ and $\psi = A\phi$ where $A \in \{\partial_\alpha\}$, by the form of the data, we clearly have

$$\|\phi\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta^{3/2}, \quad \|\psi\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta^{1/2}. \quad (3.1)$$

We will use Z or Z_j to denote any vector from $\{T, R_i, Q\}$ where $Q = t\underline{L}$. On Σ_{-2}^δ , Z is simply ∂_r , Ω_i or $-r_0(\partial_t - \partial_r)$, therefore, we have $\|Z_1 \circ Z_2 \circ \dots \circ Z_m(\psi)\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta^{1/2-l}$ where l is the number of T 's appearing in $\{Z_j\}_{1 \leq j \leq m}$. We shall use the following schematic expression

$$\|Z^m\psi\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta^{1/2-l}, \quad (3.2)$$

with l is the number of T 's and $Z \in \{T, \Omega_i, Q\}$. We remark that in this paper $l \leq 2$ and Q appears at most twice in the string of Z 's.

We also consider the incoming energy for $Z^m \psi$ on Σ_{-2}^δ . According to (3.2), we have

$$\|\underline{L}(Z^m \psi)\|_{L^2(\Sigma_{-2}^\delta)} + \|\not{d}(Z^m \psi)\|_{L^2(\Sigma_{-2}^\delta)} \lesssim \delta^{1-l}, \quad \|T(Z^m \psi)\|_{L^2(\Sigma_{-2}^\delta)} \lesssim \delta^{-l}$$

where \not{d} denotes for the exterior differential on $S_{t,u}$. In terms of L , for $m \in \mathbb{Z}_{\geq 0}$, we obtain

$$\|\underline{L}(Z^m \psi)\|_{L^2(\Sigma_{-2}^\delta)} + \|\not{d}(Z^m \psi)\|_{L^2(\Sigma_{-2}^\delta)} \lesssim \delta^{1-l}, \quad \|L(Z^m \psi)\|_{L^2(\Sigma_{-2}^\delta)} \lesssim \delta^{-l} \quad (3.3)$$

where l is the number of T 's in Z 's.

We also consider the estimates on some connection coefficients on Σ_{-r_0} . For μ , since we have $g(T, T) = c^{-2}\mu^2$ and $T = \partial_r$ on Σ_{-r_0} , we then have $\mu = c$ on Σ_{r_0} . Since $c = (1 + 2G''(0)(\partial_t \phi)^2)^{-\frac{1}{2}}$, according to (3.1), for sufficiently small δ , we obtain

$$\|\mu - 1\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta. \quad (3.4)$$

For $\underline{\chi}_{AB}$, since $\underline{\chi}_{AB} = -c\theta_{AB} = -\frac{c}{r_0}\not{g}_{AB}$, we have $\underline{\chi}_{AB} + \frac{1}{r_0}\not{g}_{AB} = (1-c)\frac{1}{r_0}\not{g}_{AB}$. Hence,

$$\|\underline{\chi}_{AB} + \frac{1}{r_0}\not{g}_{AB}\|_{L^\infty(\Sigma_{-2}^\delta)} \lesssim \delta. \quad (3.5)$$

It measures the difference between the 2nd fundamental form with respect to $g_{\alpha\beta}$ and $m_{\alpha\beta}$.

3.2. Bootstrap assumptions and the main estimates. We expect the estimates (3.1), (3.2) and (3.3) hold not only for $t = -2$ but also for later time slice in W_δ^* . For this purpose, we will run a bootstrap argument to derive the *a priori* estimates for the $Z^m \psi$'s.

3.2.1. Conventions. We first introduce three large positive integers N_{top} , N_μ and N_∞ . They will be determined later on. We require that $N_\mu = \lfloor \frac{3}{4}N_{\text{top}} \rfloor$ and $N_\infty = \lfloor \frac{1}{2}N_{\text{top}} \rfloor + 1$. N_{top} will eventually be the total number of derivatives applied to the linearized equation $\square_{\bar{g}}\psi = 0$.

To count the number of derivatives, we define the *order* of an object. The solution ϕ is considered as an order -1 object. The variations $\psi = A\phi$ are of order 0. The metric g depends only on ψ , so it is of order 0. The inverse density function μ is of order 0. The connection coefficients are 1st order derivatives on g , hence, of order 1. In particular, $\underline{\chi}_{AB}$ is of order 1. Let $\alpha = (i_1, \dots, i_{k-1})$ be a multi-index with i_j 's from $\{1, 2, 3\}$. We use $Z^\alpha \psi$ as a schematic expression of $Z_{i_1} Z_{i_2} \dots Z_{i_{k-1}} \psi$. The order of $Z^\alpha \psi$ is $|\alpha|$, where $|\alpha| = k - 1$. Similarly, for any tensor of order $|\alpha|$, after taking m derivatives, its order becomes $|\alpha| + m$. The highest order objects in this paper will be of order $N_{\text{top}} + 1$.

Let $l \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$. We use \mathcal{O}_k^l or $\mathcal{O}_k^{\leq l}$ to denote any term of order l or at most l with estimates

$$\|\mathcal{O}_k^l\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{\frac{1}{2}k}, \quad \|\mathcal{O}_k^{\leq l}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{\frac{1}{2}k}.$$

Similarly, we use Ψ_k^l or $\Psi_k^{\leq l}$ to denote any term of order l or at most l with estimates

$$\|\Psi_k^l\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{\frac{1}{2}k}, \quad \|\Psi_k^{\leq l}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{\frac{1}{2}k},$$

and moreover, it can be explicitly expressed a function of the variations ψ . For example, $\partial_t \phi \cdot \partial_i \phi \in \Psi_2^0$; A term of the form $\prod_{i=1}^n Z^{\alpha_i} \psi$ so that $\max |\alpha_i| \leq m$ is $\Psi_{n-2l}^{\leq m}$, where l is the number of T appearing in the derivatives. Note that $\underline{\chi}$ and μ can not be expressed explicitly in terms of ψ . The \mathcal{O}_k^l terms (or similarly the Ψ_k^l terms) obey the following algebraic rules:

$$\mathcal{O}_k^{\leq l} + \mathcal{O}_{k'}^{\leq l'} = \mathcal{O}_{\min(k, k')}^{\leq \max(l, l')}, \quad \mathcal{O}_k^{\leq l} \mathcal{O}_{k'}^{\leq l'} = \mathcal{O}_{k+k'}^{\leq \max(l, l')}.$$

3.2.2. *Bootstrap assumptions on L^∞ norms.* Motivated by (3.2), we make the following bootstrap assumptions (B.1) on W_δ^* : For all t and $2 \leq |\alpha| \leq N_\infty$,²

$$\|\psi\|_{L^\infty(\Sigma_t^\delta)} + \|\underline{L}\psi\|_{L^\infty(\Sigma_t^\delta)} + \|\not{d}\psi\|_{L^\infty(\Sigma_t^\delta)} + \delta\|T\psi\|_{L^\infty(\Sigma_t^\delta)} + \delta^l\|Z^\alpha\psi\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{\frac{1}{2}}M. \quad (\text{B.1})$$

where l is the number of T 's appearing in Z^α and M is a large positive constant depending on ϕ . We will show that if δ is sufficiently small which may depend on M , then we can choose M in such a way that it depends only on the initial datum.

3.2.3. *Energy norms.* Let $d\mu_\not{g}$ be the volume form of \not{g} , for a function $f(t, \underline{u}, \theta)$, we define

$$\int_{\Sigma_t^{\underline{u}}} f = \int_0^{\underline{u}} \left(\int_{S_{t, \underline{u}'}} f(t, \underline{u}', \theta) d\mu_{\not{g}} \right) d\underline{u}', \quad \int_{\underline{C}_{\underline{u}}^t} f = \int_{-r_0}^t \left(\int_{S_{\tau, \underline{u}}} f(\tau, \underline{u}, \theta) d\mu_{\not{g}} \right) d\tau.$$

For a function $\Psi(t, \underline{u}, \theta)$, we define the energy flux through the hypersurfaces $\Sigma_t^{\underline{u}}$ and $\underline{C}_{\underline{u}}^t$ as

$$\begin{aligned} E(\Psi)(t, \underline{u}) &= \int_{\Sigma_t^{\underline{u}}} (L\Psi)^2 + \mu^2 |\not{d}\Psi|^2, \quad F(\Psi)(t, \underline{u}) = \int_{\underline{C}_{\underline{u}}^t} \mu |\not{d}\Psi|^2, \\ \underline{E}(\Psi)(t, \underline{u}) &= \int_{\Sigma_t^{\underline{u}}} \mu (\underline{L}\Psi)^2 + \mu |\not{d}\Psi|^2, \quad \underline{F}(\Psi)(t, \underline{u}) = \int_{\underline{C}_{\underline{u}}^t} (\underline{L}\Psi)^2. \end{aligned} \quad (3.6)$$

For each integer $0 \leq k \leq N_{\text{top}}$, we define

$$\begin{aligned} E_{k+1}(t, \underline{u}) &= \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} E(Z^{\alpha+1}\psi)(t, \underline{u}), \quad F_{k+1}(t, \underline{u}) = \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} F(Z^{\alpha+1}\psi)(t, \underline{u}), \\ \underline{E}_{k+1}(t, \underline{u}) &= \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} \underline{E}(Z^{\alpha+1}\psi)(t, \underline{u}), \quad \underline{F}_{k+1}(t, \underline{u}) = \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} \underline{F}(Z^{\alpha+1}\psi)(t, \underline{u}), \end{aligned} \quad (3.7)$$

where l is the number of T 's appearing in Z^α . The symbol \sum_{ψ} means to sum over all the first order variations $A\phi$ of ψ . For the sake of simplicity, we shall omit this sum symbol in the sequel.

For each integer $0 \leq k \leq N_{\text{top}}$, we assign a nonnegative integer b_k to k in such a way that

$$b_0 = b_1 = \dots = b_{N_\mu} = 0, \quad b_{N_\mu+1} < b_{N_\mu+2} < \dots < b_{N_{\text{top}}}. \quad (3.8)$$

We call b_k 's the *blow-up indices*. The sequence $(b_k)_{0 \leq k \leq N_{\text{top}}}$ will be determined later on.

For each integer $0 \leq k \leq N_{\text{top}}$, we also define the modified energy $\tilde{E}_k(t, \underline{u})$ and $\tilde{\underline{E}}_k(t, \underline{u})$ as

$$\begin{aligned} \tilde{E}_{k+1}(t, \underline{u}) &= \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} E_{k+1}(\tau, \underline{u}) \}, \quad \tilde{\underline{E}}_{k+1}(t, \underline{u}) = \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} \underline{E}_{k+1}(\tau, \underline{u}) \} \\ \tilde{F}_{k+1}(t, \underline{u}) &= \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} F_{k+1}(\tau, \underline{u}) \}, \quad \tilde{\underline{F}}_{k+1}(t, \underline{u}) = \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} \underline{F}_{k+1}(\tau, \underline{u}) \} \end{aligned} \quad (3.9)$$

We now state the main estimates of the paper.

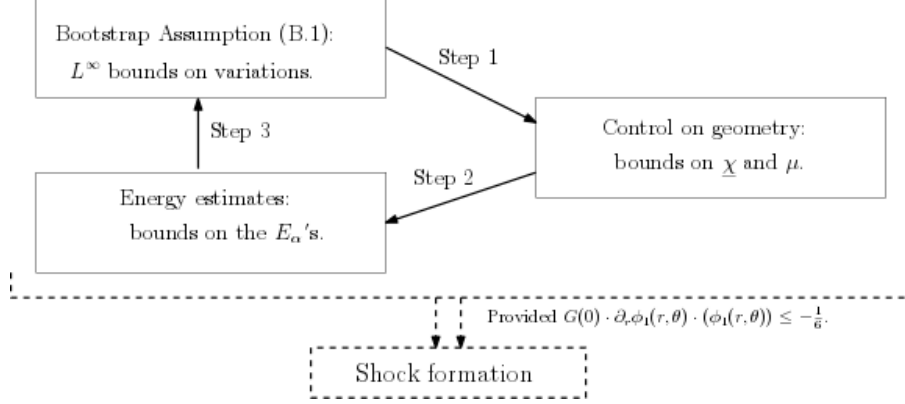
Theorem 3.1. *There exists a constant δ_0 depending only on the seed data ϕ_0 and ϕ_1 , so that for all $\delta < \delta_0$, there exist constants M_0 , N_{top} and $(b_k)_{0 \leq k \leq N_{\text{top}}}$ with the following properties*

- M_0 , N_{top} and $(b_k)_{0 \leq k \leq N_{\text{top}}}$ depend only on the initial datum.
- The inequalities (B.1) holds for all $t < t^*$ with $M = M_0$.
- Either $t^* = -1$ and we have a smooth solution in the time slab $[-2, -1]$; or $t^* < -1$ and then ψ_α 's as well as the rectangular coordinates x^i 's extend smoothly as functions of the coordinates $(t, \underline{u}, \theta)$ to $t = t^*$ and there is at least one point on $\Sigma_{t^*}^\delta$ where μ vanishes, thus we have shock formation.

²For a multi-index α , the symbol $\alpha - 1$ means another multi-index β with degree $|\beta| = |\alpha| - 1$.

- If, moreover, the initial data satisfies the largeness condition (1.6), then in fact $t^* < -1$.

Before we start the detailed analysis, it is instructional to provide a 3-step scheme to illustrate the structure of the proof (of Theorem 3.1):



3.3. Preliminary results based on (B.1).

3.3.1. *Estimates on metric and connection.* We start with bounds on c .

Lemma 3.2. *For sufficiently small³ δ , we have*

$$\begin{aligned} \|c - 1\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta M^2, \quad \frac{1}{2} \leq c \leq 2, \\ \|\underline{L}c\|_{L^\infty(\Sigma_t^\delta)} + \|\not{D}c\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta M^2, \quad \|Tc\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2. \end{aligned} \quad (3.10)$$

Proof. Let $\psi_0 = \partial_t \phi$. Since $c = (1 + 3G''(0)(\partial_t \phi)^2)^{-\frac{1}{2}}$, in view of (B.1), we can take $\varepsilon = \frac{1}{4G''(0)M^2}$. Therefore, the quantity in the parenthesis falls in $[\frac{1}{4}, \frac{7}{4}]$, this implies the bound on c .

On the other hand, $\underline{L}c = -(3/2)G''(0)(1 + 3G''(0)\psi_0^2)^{-\frac{3}{2}}\psi_0 \cdot \underline{L}\psi_0$. We then use (B.1) exactly in the same way as above, this gives the bound on $\underline{L}c$. Similarly, we can obtain other bounds in (3.10). \square

We now derive estimates on m , e and μ :

Lemma 3.3. *For sufficiently small δ , we have*

$$\|m\|_{L^\infty(\Sigma_t^\delta)} + \|\not{D}m\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \quad \|Tm\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{-1}M^2, \quad (3.11)$$

$$\|e\|_{L^\infty(\Sigma_t^\delta)} + \|\not{D}e\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^2, \quad \|Te\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \quad (3.12)$$

$$\|\mu\|_{L^\infty(\Sigma_t^\delta)} + \|\underline{L}\mu\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2. \quad (3.13)$$

Proof. Let $\psi_0 = \partial_t \phi$. We first bound m . Since $m \sim (1 \pm \psi_0^2)^{-2}\psi_0 \cdot T\psi_0$, according to (B.1), we have

$$\|m\|_{L^\infty(\Sigma_t^\delta)} \lesssim \frac{1}{(1 - 3G''(0)\|\psi_0\|_{L^\infty(\Sigma_t^\delta)}^2)^2} \|\psi_0\|_{L^\infty(\Sigma_t^\delta)} \cdot \|T\psi_0\|_{L^\infty(\Sigma_t^\delta)} \lesssim \frac{1}{(1 - 3G''(0)\delta M^2)^2} \cdot \delta^{\frac{1}{2}}M \cdot \delta^{-\frac{1}{2}}M.$$

This clearly implies the bound on m for sufficiently small δ . For the derivatives of m , say Tm , it is based on (B.1) and the explicit computation $Tm = \frac{3G''(0)}{(1+3G''(0)\psi_0^2)^2}\psi_0 \cdot T^2\psi_0 + \frac{3G''(0)}{(1+3G''(0)\psi_0^2)^2}(T\psi_0)^2 - \frac{36G''(0)^2}{(1+3G''(0)\psi_0^2)^3}\psi_0^2(T\psi_0)^2$. Similarly, we can bound $\not{D}m$.

³This sentence always means that, there exists $\varepsilon = \varepsilon(M)$ so that for all $\delta \leq \varepsilon$, we have ...

The estimates on e can be derived exactly in the same way, so we omit the proof.

To bound μ , we integrate the equation $\underline{L}\mu = m + \mu e$ to derive

$$\mu(t, \underline{u}, \theta) = \exp\left(\int_{-r_0}^t e(\tau) d\tau\right) \mu(-r_0, \underline{u}, \theta) + \int_{-r_0}^t \exp\left(\int_{\tau}^t e(\tau') d\tau'\right) m(\tau, \underline{u}, \theta) d\tau. \quad (3.14)$$

Then (3.4), (3.11) and (3.12) imply the estimate on μ immediately. For $\underline{L}\mu$, we simply combine $\underline{L}\mu = m + \mu e$, (3.11), (3.12) and the bound on μ . This completes the proof. \square

We move to the bounds on $T\mu$ and $\not{d}\mu$:

Lemma 3.4. *For sufficiently small δ , we have*

$$\|T\mu\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{-1} M^2, \quad \|\not{d}\mu\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2. \quad (3.15)$$

Proof. The idea is to commute T and \not{d} with $\underline{L}\mu = m + \mu e$. For $\not{d}\mu$, we have $\not{d}\underline{L}(\not{d}\mu) = e\not{d}\mu + (\not{d}m + \mu\not{d}e)$. We have already seen that $\|\not{d}m\|_{L^\infty(\Sigma_t^\delta)} + \|\mu\not{d}e\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2$. Therefore, we integrate along \underline{L} to derive the desired bound on $\not{d}\mu$.

For $T\mu$, we have $\underline{L}(T\mu) = eT\mu + (Tm + \mu Te - (\underline{\zeta}^A + \underline{\eta}^A)\not{d}_A\mu)$. We first show that

$$\|\underline{\zeta}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^2, \quad \|\underline{\eta}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2. \quad (3.16)$$

In fact, $\underline{\zeta}_A = -c^{-1}\mu\not{d}_A(c)$. The bound on $\underline{\zeta}$ the follows immediately from (3.10) and (3.13). Since $\underline{\eta} = \underline{\zeta} + \not{d}\mu$, the bound on $\underline{\eta}$ is also clear because we have just obtained estimates on $\not{d}\mu$.

Back to the formula for $\underline{L}(T\mu)$, for small δ , we bound the terms in the parenthesis by

$$\|(Tm + \mu Te - (\underline{\zeta}^A + \underline{\eta}^A)\not{d}_A\mu)\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta^{-1} M^2.$$

We then integrate to derive the bound for $T\mu$. This completes the proof. \square

We now estimate $\underline{\chi}_{AB}$. For this purpose, we introduce

$$\underline{\chi}'_{AB} = \underline{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u} - t}. \quad (3.17)$$

which measures the deviation of $\underline{\chi}_{AB}$ from the null 2nd fundamental form in Minkowski space. We have

Lemma 3.5. *For sufficiently small δ , we have*

$$\|\underline{\chi}'_{AB}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^2. \quad (3.18)$$

Proof. According to (2.37), we have

$$\underline{L}(\underline{\chi}'_{AB}) = e\underline{\chi}'_{AB} + \underline{\chi}'^C_A \underline{\chi}'_{BC} - \frac{e}{\underline{u} - t} \not{g}_{AB} - \underline{\alpha}'_{AB}. \quad (3.19)$$

Hence, $\underline{L}|\underline{\chi}'|^2_{\not{g}} = 2e|\underline{\chi}'|^2 - 2\underline{\chi}'^B_A \underline{\chi}'^C_B \underline{\chi}'^A_C + \frac{2|\underline{\chi}'|^2}{\underline{u} - t} - \frac{2e}{\underline{u} - t} \text{tr}\underline{\chi}' - 2\underline{\chi}'^{AB} \underline{\alpha}'_{AB}$. Therefore, we obtain

$$\underline{L}((t - \underline{u})^2 |\underline{\chi}'|) \lesssim (t - \underline{u})^2 \left((|e| + |\underline{\chi}'|) |\underline{\chi}'| + \frac{|e|}{\underline{u} - t} + |\underline{\alpha}'| \right) \quad (3.20)$$

where all norms are defined with respect to \not{g} .

Let $\mathcal{P}(t)$ be the property that $\|\underline{\chi}'\|_{L^\infty(\Sigma_t^\delta)} \leq C_0 \delta M^2$ for all $t' \in [-r_0, t]$. By choosing C_0 suitably large, according to the assumptions on initial data, we have $\|\underline{\chi}'\|_{L^\infty(\Sigma_{-2}^\delta)} < C_0 \delta$. It follows by continuity that $\mathcal{P}(t)$ is true for t sufficiently close to $-r_0$. Let t_0 be the upper bound of $t \in [-r_0, t_0]$ for which $\mathcal{P}(t)$ holds. By continuity, $\mathcal{P}(t_0)$ is true. Therefore, for $t \leq t_0$, we have $|\underline{\chi}'| + |e| \leq (C_0 + C_1) \delta M^2$ for a universal

The deformation tensor of $Z_5 = Q$ is given by

$$\begin{aligned} {}^{(Q)}\tilde{\pi}_{\underline{LL}} &= 0, \quad {}^{(Q)}\tilde{\pi}_{LL} = 4t\underline{L}(c^{-2}\mu)(c^{-1}\mu) - 4c^{-3}\mu^2, \quad {}^{(Q)}\tilde{\pi}_{L\underline{L}} = -2t\underline{L}(c^{-1}\mu) - 2c^{-1}\mu \\ {}^{(Q)}\tilde{\pi}_{\underline{LA}} &= 0, \quad {}^{(Q)}\tilde{\pi}_{LA} = 2tc^{-1}(\underline{\zeta}_A + \underline{\eta}_A), \quad {}^{(Q)}\hat{\tilde{\pi}}_{AB} = 2tc^{-1}\hat{\underline{\chi}}_{AB}, \quad \text{tr} {}^{(Q)}\tilde{\pi} = 2c^{-1}t\text{tr}_{\tilde{g}}\tilde{\chi}. \end{aligned} \quad (3.26)$$

By virtue of the estimates already derived, for sufficiently small δ , we have

$$\begin{aligned} \|\mu^{-1} {}^{(Q)}\tilde{\pi}_{LL}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim M^2, \quad \|{}^{(Q)}\tilde{\pi}_{LL}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \quad \|{}^{(Q)}\tilde{\pi}_{LA}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \\ \|\hat{{}^{(Q)}\tilde{\pi}}_{AB}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta M^2, \quad \|\text{tr} {}^{(Q)}\tilde{\pi}\|_{L^\infty(\Sigma_t^\delta)} \lesssim 1. \end{aligned} \quad (3.27)$$

Actually the estimate for $\text{tr} {}^{(Q)}\tilde{\pi}$ can be improved more precisely. By (3.27) : $\|\text{tr} {}^{(Q)}\tilde{\pi}\|_{L^\infty(\Sigma_t^\delta)} \lesssim 1$ and the equation (4.1) together with the bootstrap assumptions, we see the order of the former is also 1. While the estimate for the latter is a bit more delicate. Let us rewrite the following component of deformation tensor of Q :

$$\text{tr} {}^{(Q)}\tilde{\pi} = 2c^{-1}t\text{tr}_{\tilde{g}}\tilde{\chi} = 2c^{-1}t\text{tr}_{\tilde{g}}\tilde{\chi}' + 4\frac{c^{-1}\underline{u}}{\underline{u}-t} - 4(c^{-1}-1) - 4$$

This tells us:

$$\|\text{tr} {}^{(Q)}\tilde{\pi} + 4\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta. \quad (3.28)$$

We study the deformation tensors of R_i 's. Based on (2.42), we compute ${}^{(R_i)}\pi_{\alpha\beta}$ with respect to g :

$$\begin{aligned} {}^{(R_i)}\pi_{\underline{LL}} &= 0, \quad {}^{(R_i)}\pi_{TT} = 2c^{-1}\mu \cdot R_i(c^{-1}\mu), \quad {}^{(R_i)}\pi_{\underline{LT}} = -R_i(\mu), \quad {}^{(R_i)}\pi_{AB} = -2\lambda_i\theta_{AB}, \\ {}^{(R_i)}\pi_{TA} &= -c^{-1}\mu(\theta_{AB} - \frac{\not{g}_{AB}}{\underline{u}-t})R_i^B + c^{-1}\mu\varepsilon_{ikj}y^k X_A^j + \lambda_i X_A(c^{-1}\mu), \\ {}^{(R_i)}\pi_{\underline{LA}} &= -\underline{\chi}_{AB}R_i^B + \underline{L}^k\varepsilon_{ikj}X_A^j + c\mu^{-1}\lambda_i\underline{\zeta}_A = -(\underline{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u}-t})R_i^B + \varepsilon_{ikj}z^k X_A^j + c\mu^{-1}\lambda_i\underline{\zeta}_A. \end{aligned} \quad (3.29)$$

The Latin indices i, j, k are defined with respect to the Cartesian coordinates on Σ_t . To bound deformation of R_i , it suffices to control the λ_i 's, y^i 's and z^i 's.

First of all, we have

$$|\hat{T}^i| + |\underline{L}^i| \lesssim 1. \quad (3.30)$$

The proof is straightforward: $g|_{\Sigma_t^\delta}$ is flat and \hat{T} is the unit normal of $S_{t,\underline{u}}$ in Σ_t^δ , so $|\hat{T}^i| \leq 1$. In the Cartesian coordinates (t, x^1, x^2, x^3) , $\underline{L} = \partial_t - c\hat{T}^i\partial_i$, so $|\underline{L}^i| \lesssim 1$.

Let $r = (\sum_{i=1}^3 x^i)^{\frac{1}{2}}$. Since $Tr = c^{-1}\mu \sum_{i=1}^3 \frac{x^i \hat{T}^i}{r}$, (3.30) implies that $|Tr| \lesssim M^2$. We then integrate from 0 to \underline{u} , since $r = -t$ when $\underline{u} = 0$ and $|\underline{u}| \leq \delta$, we obtain $|r+t| \lesssim \delta M^2$. In application, for sufficiently small δ , we often use $r \sim |t|$. The estimate can also be written as

$$\left| \frac{1}{r} - \frac{1}{\underline{u} + |t|} \right| \lesssim \delta M^2. \quad (3.31)$$

To control λ_i , we consider its \underline{L} derivative. By definition $\lambda_i = g(\Omega_i, \hat{T})$, we can write its derivative along \underline{L} as $\underline{L}\lambda_i = \sum_{k=1}^3 (\Omega_i)^k \underline{L}\hat{T}^k = \sum_{k=1}^3 (\Omega_i)^k X^A(c)X_A^k$. As $|t| \sim r$, we have $|\Omega_i| \lesssim |t| \lesssim 1$, this implies

$$\|\underline{L}\lambda_i\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^2. \quad (3.32)$$

Since $\lambda_i = 0$ on $\Sigma_{-r_0}^\delta$, we have

$$\|\lambda_i\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^2. \quad (3.33)$$

To control y^i 's and z^i 's, let $\bar{y} = (y^1, y^2, y^3)$ and $\bar{x} = (x^1, x^2, x^3)$, we then have

$$|\bar{y} - (\frac{1}{r} - \frac{1}{\underline{u}-t})\bar{x}|^2 = |(g(\widehat{T}, \partial_r) - 1)\partial_r|^2 + \frac{1}{r^2} \sum_{i=1}^3 \lambda_i^2 \quad (3.34)$$

On the other hand, we have $1 - |g(\widehat{T}, \partial_r)|^2 = \frac{1}{r^2} \sum_{i=1}^3 \lambda_i^2 \lesssim \delta M^2$. While on $S_{t,0}$, Since $g(\partial_r, \widehat{T}) = 1$ on $S_{t,0}$, for sufficiently small δ , the angle between ∂_r and \widehat{T} is less than $\frac{\pi}{2}$, which implies $1 + g(\widehat{T}, \partial_r) \geq 1$. Therefore,

$$|1 - g(\widehat{T}, \partial_r)| \lesssim \delta M^2. \quad (3.35)$$

Together with (3.33) and (3.34), this implies

$$|y^i| \lesssim \delta M^2, \quad |y'^i| \lesssim \delta M^2. \quad (3.36)$$

We then control z^i from its definition

$$|z^i| \lesssim \delta M^2. \quad (3.37)$$

The derivatives of λ_i on Σ_t are given by $X_A(\lambda_i) = \left(\theta_{AB} - \frac{\mathcal{G}_{AB}}{\underline{u}-t}\right) R_i^B - \varepsilon_{ikj} y^k X_A^j$ and $T(\lambda_i) = -R_i(c^{-1}\mu)$. Hence,

$$\|\mathcal{D}\lambda_i\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^2, \quad \|T\lambda_i\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2$$

Finally, we obtain the following estimates for the deformation tensor of R_i :⁵

$$\begin{aligned} (R_i)\pi_{\underline{L}\underline{L}} &= 0, \quad \|(R_i)\widetilde{\pi}_{\underline{L}T}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \quad \|\mu^{-1(R_i)}\pi_{TT}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \\ \|(R_i)\pi_{\underline{L}A}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta M^2, \quad \|(R_i)\pi_{TA}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^4, \\ \|(R_i)\widehat{\pi}_{AB}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta^2 M^4, \quad \|\text{tr}^{(R_i)}\pi\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^4. \end{aligned}$$

We use the relation $L = c^{-2}\mu\underline{L} + 2T$ to rewrite the above estimates in null frame as follows:

$$\begin{aligned} \|(R_i)\pi_{\underline{L}\underline{L}}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim M^2, \quad \|\mu^{-1(R_i)}\pi_{\underline{L}\underline{L}}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \\ \|(R_i)\pi_{\underline{L}A}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta M^2, \quad \|(R_i)\pi_{LA}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^4, \\ \|(R_i)\widehat{\pi}_{AB}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta^2 M^4, \quad \|\text{tr}^{(R_i)}\pi\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^4. \end{aligned} \quad (3.38)$$

The deformation tensors of R_i 's with respect to \widetilde{g} are estimated by

$$\begin{aligned} (R_i)\widetilde{\pi}_{\underline{L}\underline{L}} &= 0, \quad \|(R_i)\widetilde{\pi}_{\underline{L}\underline{L}}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \quad \|\mu^{-1(R_i)}\widetilde{\pi}_{\underline{L}\underline{L}}\|_{L^\infty(\Sigma_t^\delta)} \lesssim M^2, \\ \|(R_i)\widetilde{\pi}_{\underline{L}A}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta M^2, \quad \|(R_i)\widetilde{\pi}_{LA}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^4, \\ \|(R_i)\widehat{\pi}_{AB}\|_{L^\infty(\Sigma_t^\delta)} &\lesssim \delta^2 M^4, \quad \|\text{tr}^{(R_i)}\widetilde{\pi}\|_{L^\infty(\Sigma_t^\delta)} \lesssim \delta M^4. \end{aligned} \quad (3.39)$$

3.3.3. Applications. As in [5] and [8], we are able to show that the R_i derivatives are equivalent to the \mathcal{D} and ∇ derivative. For a 1-form ξ on $S_{t,\underline{u}}$, we have $\sum_{i=1}^3 \xi(R_i)^2 = r^2(|\xi|^2 - (\xi(y'))^2)$. This is indeed can be derived from the formula $\sum_{i=1}^3 (R_i)^a (R_i)^b = r^2(\delta_{cd} - y'^c y'^d) \Pi_c^a \Pi_d^b$, where $a, b, c, d \in \{1, 2, 3\}$. In view of (3.31), (3.36) and the definition of y'^i , for sufficiently small δ , we have $\sum_{i=1}^3 \xi(R_i)^2 \sim r^2|\xi|^2$. Since r is bounded below and above by a constant, we obtain $\sum_{i=1}^3 \xi(R_i)^2 \sim |\xi|^2$. Similarly, for a k -covariant tensor ξ on $S_{t,\underline{u}}$, we have $\sum_{i_1, i_2, \dots, i_k=1}^3 \xi(R_{i_1}, R_{i_2}, \dots, R_{i_k})^2 \sim |\xi|^2$. In particular, we can take $\xi = \mathcal{D}\psi$, therefore, $\sum_{i=1}^3 (R_i\psi)^2 \sim |\mathcal{D}\psi|^2$. Henceforth, we omit the summation and write schematically as $|R_i\psi| \sim |\mathcal{D}\psi|$.

⁵We emphasize that the traceless part of $\widehat{\pi}_{AB}$ is defined with respect to \mathcal{G} .

We can also compare the R_i -derivatives with the ∇ -derivatives for tensors. For $S_{t,\underline{u}}$ -tangential 1-form ξ and vectorfield X , let $\mathcal{L}_{R_i}\xi$ be the orthogonal projection of the Lie derivative $\mathcal{L}_{R_i}\xi$ onto the surface $S_{t,\underline{u}}$. Since $(\mathcal{L}_{R_i}\xi)(X) = (\nabla_{R_i}\xi)(X) + \xi(\nabla_X R_i)$, we obtain

$$\sum_{i=1}^3 |\mathcal{L}_{R_i}\xi|^2 = \sum_{i=1}^3 |\xi(R_i)|^2 + 2 \sum_{i=1}^3 \xi^k (\nabla_{R_i}\xi)_a (\nabla R_i)^a{}_k + \sum_{i=1}^3 \xi^k \xi^l (\nabla R_i)_{ak} (\nabla R_i)^a{}_l.$$

We also have $\sum_{i=1}^3 |\nabla_{R_i}\xi|^2 = r^2(\delta^{cd} - y'^c y'^d)(\nabla\xi)_{ca}(\nabla\xi)_d{}^a$. In view of the estimates on y'^i , for sufficiently small δ , we obtain

$$\sum_{i=1}^3 |\nabla_{R_i}\xi|^2 \gtrsim |\nabla\xi|^2.$$

Let ε_{ijk} be the volume form on Σ_t and v_i be a $S_{t,\underline{u}}$ 1-form with rectangular components $(v_i)_a = \Pi_a^b \varepsilon_{ibk} \xi_k$. By virtue of the formula $(\nabla R_i)_l^k = \Pi_k^m \Pi_l^n \varepsilon_{inm} - \lambda_i \theta_{kl}$, we have

$$\sum_{i=1}^3 \xi^k \xi^l (\nabla R_i)_{ak} (\nabla R_i)^a{}_l = |\xi|^2 + 2c \sum_{i=1}^3 \lambda_i \xi \cdot \underline{\chi} \cdot v_i + c^2 |\underline{\chi} \cdot \xi|^2 \sum_{i=1}^3 \lambda_i^2.$$

In view of the estimates on λ_i , for sufficiently small δ , we have $\sum_{i=1}^3 \xi^k \xi^l (\nabla R_i)_{ak} (\nabla R_i)^a{}_l \sim |\xi|^2$. Similarly, we have $|\sum_{i=1}^3 \xi^k (\nabla_{R_i}\xi)_a (\nabla R_i)^a{}_k| \lesssim |\xi| |\nabla\xi|$. Finally, we conclude that

$$|\xi|^2 + |\nabla\xi|^2 \lesssim \sum_{i=1}^3 |\mathcal{L}_{R_i}\xi|^2 \lesssim |\xi|^2 + |\nabla\xi|^2.$$

Henceforth, we omit the summation and write schematically as $|\mathcal{L}_{R_i}\xi| \sim |\xi| + |\nabla\xi|$. Similarly, for a tracefree symmetric 2-tensors θ_{AB} tangential to $S_{t,\underline{u}}$, we have $|\theta| + |\nabla\theta| \lesssim |\mathcal{L}_{R_i}\theta| \lesssim |\theta| + |\nabla\theta|$. This will be applied to $\theta = \hat{\chi}_{AB}$ later on.

Another application of the pointwise estimates based on the bootstrap assumption is to give an estimate on $\sqrt{\det \underline{g}}$. On Σ_{-2}^δ , $S_{-2,\underline{u}}$ is around sphere and $\sqrt{\det \underline{g}}$ is bounded above and below by positive absolute constants. On the other hand, (3.17) and (3.18) implies that $\text{tr}\underline{\chi}$ is bounded. This together with the formula

$$\underline{L} \left(\log \sqrt{\det \underline{g}} \right) = \text{tr}\underline{\chi} \quad (3.40)$$

gives the fact that $\sqrt{\det \underline{g}}$ is bounded and never vanishes along each null generator.

3.3.4. Sobolev inequalities and elliptic estimates. To obtain the Sobolev inequalities on $S_{t,\underline{u}}$, we introduce

$$I(t, \underline{u}) = \sup_{\substack{U \in S_{t,\underline{u}}, \\ \partial U \text{ is } C^1}} \frac{\min(|U|, |S_{t,\underline{u}} - U|)}{|\partial U|^2}$$

the isoperimetric constant on $S_{t,\underline{u}}$, where $|U|$, $|S_{t,\underline{u}} - U|$ and $|\partial U|$ are the measures of the corresponding sets with respect to \underline{g} on $S_{t,\underline{u}}$. Therefore, in view of the fact that $R_i \sim \nabla$, for sufficiently small δ , we have the following Sobolev inequalities:

$$\begin{aligned} \|f\|_{W^{1,4}(S_{t,\underline{u}})} &\lesssim |I(t, \underline{u})|^{\frac{1}{4}} |S_{t,\underline{u}}|^{-\frac{3}{4}} (\|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})} + \|R_i R_j f\|_{L^2(S_{t,\underline{u}})}), \\ \|f\|_{L^\infty(S_{t,\underline{u}})} &\lesssim |I(t, \underline{u})|^{\frac{3}{4}} |S_{t,\underline{u}}|^{-\frac{1}{2}} (\|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})} + \|R_i R_j f\|_{L^2(S_{t,\underline{u}})}). \end{aligned} \quad (3.41)$$

where $\|f\|_{W^{1,4}(S_{t,\underline{u}})}$ is defined as $\|f\|_{W^{1,4}(S_{t,\underline{u}})} = |S_{t,\underline{u}}|^{-1/2} \|f\|_{L^4(S_{t,\underline{u}})} + \|\nabla f\|_{L^4(S_{t,\underline{u}})}$. It remains to control the isoperimetric constant $I(t, \underline{u})$.

We use T to generate a diffeomorphism of $S_{t,\underline{u}}$ to $S_{t,0}$ which maps U , $S_{t,\underline{u}} - U$ and ∂U to corresponding sets $U_{\underline{u}}$, $S_{t,0} - U_{\underline{u}}$ and $\partial U_{\underline{u}}$ on $S_{t,0}$. Let $U_{\underline{u}'}$, $S_{t,\underline{u}'} - U_{\underline{u}'}$, $\partial U_{\underline{u}'}$ be the inverse images of these on each $S_{t,\underline{u}'}$ for $\underline{u}' \in [0, \underline{u}]$. Since $\mathcal{L}_T \mathcal{G}_{AB} = 2c^{-1}\mu\theta = -2c^{-2}\mu\chi_{AB}$, for $\underline{u}' \in [0, \underline{u}]$, we obtain

$$\frac{d}{d\underline{u}'}(|U_{\underline{u}'}|) = - \int_{U_{\underline{u}'}} c^{-2}\mu \text{tr} \chi d\mu_{\mathcal{G}}, \quad \frac{d}{d\underline{u}'}(|\partial U_{\underline{u}'}|) = - \int_{U_{\underline{u}'}} c^{-2}\mu \chi(\nu, \nu) ds,$$

where ν is the unit normal of $\partial U_{\underline{u}'}$ in $S_{t,0}$ and ds the element of arc length of $\partial U_{\underline{u}'}$. In view of the estimates on χ and μ derived before, for sufficiently small δ , we have

$$\frac{d}{d\underline{u}'}(|U_{\underline{u}'}|) \lesssim |U_{\underline{u}'}|, \quad \frac{d}{d\underline{u}'}(|\partial U_{\underline{u}'}|) \lesssim |\partial U_{\underline{u}'}|.$$

Therefore, by integrating from 0 to \underline{u} , we have

$$|U| \sim |U_{\underline{u}}|, \quad |\partial U| \sim |\partial U_{\underline{u}}|.$$

Hence, $I(t, \underline{u}) \sim I(t, 0) \sim 1$. Finally, since $|S_{t,\underline{u}}| \sim 1$ (This is seen by the fact $d\mu_{\mathcal{G}(t,\underline{u})} \sim d\mu_{\mathcal{G}(-r_0,0)}$, which can be shown by calculations in [5].), we conclude that

$$\|f\|_{W^{1,4}(S_{t,\underline{u}})} + \|f\|_{L^\infty(S_{t,\underline{u}})} \lesssim \|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})} + \|R_i R_j f\|_{L^2(S_{t,\underline{u}})}. \quad (3.42)$$

We remark that, similarly, we have

$$\|f\|_{L^4(S_{t,\underline{u}})} \lesssim \|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})}. \quad (3.43)$$

We also have the following elliptic estimates for traceless two-tensors.

Lemma 3.7. *If δ is sufficiently small, for any traceless 2-covariant symmetric tensor θ_{AB} on $S_{t,\underline{u}}$, we have*

$$\int_{S_{t,\underline{u}}} \mu^2 (|\nabla \theta|^2 + |\theta|^2) d\mu_{\mathcal{G}} \lesssim \int_{S_{t,\underline{u}}} \mu^2 |\mathring{d}\mathring{v}\theta|^2 + |\mathring{d}\mu|^2 |\theta|^2 d\mu_{\mathcal{G}} \quad (3.44)$$

Proof. Let $J^A = \theta^B{}_C \nabla_B \theta^{AC} - \theta^A{}_C (\mathring{d}\mathring{v}\theta)^C$, then $|J| \lesssim |\theta|(|\nabla \theta| + |\mathring{d}\mathring{v}\theta|)$. The Bôchner formula says

$$|\nabla \theta|^2 + 2K|\theta|^2 = 2|\mathring{d}\mathring{v}\theta|^2 + \mathring{d}\mathring{v}J,$$

where K is the Gauss curvature. According to (2.29) and the estimates on χ_{AB} , for sufficiently small δ , we know that $|K| \sim 1$. Therefore, we have

$$|\nabla \theta|^2 + |\theta|^2 \sim 2|\mathring{d}\mathring{v}\theta|^2 + \mathring{d}\mathring{v}J,$$

We then multiply both sides by μ^2 and integrate on $S_{t,\underline{u}}$. The Cauchy-Schwarz inequality together with the above estimates on $|J|$ as well as the divergence theorem yields the inequality. \square

4. THE BEHAVIOR OF THE INVERSE DENSITY FUNCTION

The behavior of the inverse density function μ plays an dominant rôle in this paper. The method of obtaining estimates on μ is to relate μ to its initial value on $\Sigma_{t=-r_0}$. Since the metric g depends only on $\psi_0 = \partial_t \phi$, μ is also determined by ψ_0 . This leads naturally to the study of the wave equation $\square_{\bar{g}} \psi_0 = 0$. We can rewrite it in the null frame as

$$\underline{L}(L\psi_0) + \frac{1}{2} \text{tr} \tilde{\chi} \cdot L\psi_0 + (-\mu \mathring{\Delta} \psi_0 + \frac{1}{2} \text{tr} \tilde{\chi} \cdot \underline{L}\psi_0 + 2\underline{\zeta} \cdot \mathring{d}\psi_0 + \mu \mathring{d} \log(c) \cdot \mathring{d}\psi_0) = 0. \quad (4.1)$$

4.1. The asymptotic expansion for μ . We start with a lemma which relates $L\psi_0(t, \underline{u}, \theta)$ to its initial value on Σ_{-r_0} .

Lemma 4.1. *For sufficiently small δ , we have*

$$| |t|L\psi_0(t, \underline{u}, \theta) - r_0L\psi_0(-r_0, \underline{u}, \theta) | \lesssim \delta^{\frac{1}{2}}M^3. \quad (4.2)$$

Proof. We regard (4.1) as a transport equation for $L\psi_0$. According to (B.1) and the estimates from previous sections, the L^∞ norm of the terms in the big parenthesis in (4.1) is bounded by $\delta^{\frac{1}{2}}M^3$. Hence,

$$| \underline{L}(L\psi_0)(t, \underline{u}, \theta) + \frac{1}{2} \text{tr} \tilde{\chi}(t, \underline{u}, \theta) \cdot L\psi_0(t, \underline{u}, \theta) | \lesssim \delta^{\frac{1}{2}}M^3.$$

By virtue of (3.23), this implies $| \underline{L}(L\psi_0)(t, \underline{u}, \theta) - \frac{1}{\underline{u}-t} L\psi_0(t, \underline{u}, \theta) | \lesssim \delta^{\frac{1}{2}}M^3$. Therefore, we obtain

$$| \underline{L}((\underline{u}-t)L\psi_0)(t, \underline{u}, \theta) | \lesssim \delta^{\frac{1}{2}}M^3.$$

Since $|\underline{u}| \leq \delta$, we integrate from $-r_0$ to t and this yields the desired estimates. \square

Remark 4.2. *The estimates 4.2 also hold for $R_iL\psi_0$ or $R_iR_j\psi_0$, e.g., see (4.12). To derive these estimates, we commute R_i 's with (4.1) and follow the same way as in the above proof.*

Since $L = c^{-2}\mu\underline{L} + 2T$, as a corollary, we have

Corollary 4.3. *For sufficiently small δ , we have*

$$| |t|T\psi_0(t, \underline{u}, \theta) - r_0T\psi_0(-r_0, \underline{u}, \theta) | \lesssim \delta^{\frac{1}{2}}M^3, \quad (4.3)$$

$$| |t|\psi_0(t, \underline{u}, \theta) - r_0\psi_0(-r_0, \underline{u}, \theta) | \lesssim \delta^{\frac{3}{2}}M^3. \quad (4.4)$$

Proof. For (4.4), we integrate (4.3) for \underline{u}' from 0 to \underline{u} and use the fact that $\psi_0(t, 0, \theta) = 0$. \square

We turn to the behavior of $\underline{L}\mu$.

Lemma 4.4. *For sufficiently small δ , we have*

$$| |t|^2\underline{L}\mu(t, \underline{u}, \theta) - r_0^2\underline{L}\mu(-r_0, \underline{u}, \theta) | \lesssim \delta M^4. \quad (4.5)$$

Proof. According to (2.36), we write $|t|^2\underline{L}\mu(t, \underline{u}, \theta) - r_0^2\underline{L}\mu(-r_0, \underline{u}, \theta)$ as

$$(|t|^2m(t, \underline{u}, \theta) - r_0^2m(-r_0, \underline{u}, \theta)) + [|t|^2(\mu \cdot e)(t, \underline{u}, \theta) - r_0^2(\mu \cdot e)(-r_0, \underline{u}, \theta)].$$

In view of (3.12), we bound the terms in the bracket by δM^4 up to a universal constant. Therefore,

$$|t|^2\underline{L}\mu(t, \underline{u}, \theta) - r_0^2\underline{L}\mu(-r_0, \underline{u}, \theta) = (|t|^2m(t, \underline{u}, \theta) - r_0^2m(-r_0, \underline{u}, \theta)) + O(\delta M^4).$$

Since

$$|t|^2m(t, \underline{u}, \theta) - r_0^2m(-r_0, \underline{u}, \theta) = \tau^2 \frac{3G''(0)(\psi_0(\tau, \underline{u}, \theta) \cdot T\psi_0(\tau, \underline{u}, \theta))}{(1 + 3G''(0)\psi_0^2(\tau, \underline{u}, \theta))^2} \Big|_{\tau=-r_0}^{\tau=t}.$$

It is clear that the estimates follow immediately after (4.3) and (4.4). \square

We are now able to prove an accurate estimate on μ .

Proposition 4.5. *For sufficiently small δ , we have*

$$| \mu(t, \underline{u}, \theta) - 1 + r_0^2(\frac{1}{t} + \frac{1}{r_0})\underline{L}\mu(-r_0, \underline{u}, \theta) | \lesssim \delta M^4. \quad (4.6)$$

In particular, we have $\mu \leq C_0$ where C_0 is a universal constant depending only on the initial data.

Proof. According to the previous lemma, we integrate $\underline{L}\mu$:

$$\begin{aligned} \mu(t, \underline{u}, \theta) - \mu(-r_0, \underline{u}, \theta) &= \int_{-r_0}^t \underline{L}\mu(\tau, \underline{u}, \theta) d\tau = \int_{-r_0}^t \frac{\tau^2 \underline{L}\mu(\tau, \underline{u}, \theta)}{\tau^2} d\tau \\ &\stackrel{(4.5)}{=} \int_{-r_0}^t \frac{r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta)}{\tau^2} + \frac{O(\delta M^4)}{\tau^2} d\tau \\ &= -\left(\frac{1}{t} + \frac{1}{r_0}\right) r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) + O(\delta M^4). \end{aligned}$$

Therefore, we can use (3.4) to conclude. \square

We are ready to derive two key properties of the inverse density function μ . The first asserts that the shock wave region is trapping for μ , namely, once $p \in W_{shock}$, then all the points after p along the incoming null geodesic stay in W_{shock} .

Proposition 4.6. *For sufficiently small δ and for all $(t, \underline{u}, \theta) \in W_{shock}$, we have*

$$\underline{L}\mu(t, \underline{u}, \theta) \leq -\frac{1}{4|t|^2} \lesssim -1. \quad (4.7)$$

Proof. For $(t, \underline{u}, \theta) \in W_{shock}$, we have $\mu(t, \underline{u}, \theta) < \frac{1}{10}$. In view of (4.6), we claim that $r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) < 0$. Otherwise, since $\frac{1}{t} + \frac{1}{r_0} < 0$, we would have $\mu(t, \underline{u}, \theta) \geq 1 + O(\delta M^2) > \frac{1}{10}$, provided δ is sufficiently small. This contradicts the fact that $\mu(t, \underline{u}, \theta) < \frac{1}{10}$.

We can also use this argument to show that $(\frac{1}{t} + \frac{1}{r_0}) r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \geq \frac{1}{2}$. Otherwise, for sufficiently small δ , we would have $\mu(t, \underline{u}, \theta) \geq \frac{1}{2} + O(\delta M^2) > \frac{1}{10}$.

Therefore, we obtain $r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \leq \frac{1}{2} \frac{r_0 t}{r_0 + t}$. In view of (4.5), we have

$$t^2 \underline{L}\mu(t, \underline{u}, \theta) \leq \frac{1}{2} \frac{r_0 t}{r_0 + t} + O(\delta M^2).$$

Here we write M^2 instead of M^4 because we already know that $\mu \leq C_0$, where C_0 depends only on initial data. By taking a sufficiently small δ and noticing that $\frac{r_0 t}{r_0 + t}$ is bounded from above by a negative number, this yields the desired estimates. \square

Remark 4.7. *In the case when shock forms, i.e. $\mu \rightarrow 0$, by the previous proposition and (2.36), $m = -\frac{1}{2} \frac{dc^2}{d\rho} T\rho \lesssim -1$. In other words,*

$$T\rho \geq c_0 > 0$$

for some absolute constant c_0 . On the other hand, $\widehat{T}\rho = c\mu^{-1}T\rho$ and $\|\widehat{T}\| = 1$, therefore as $\mu \rightarrow 0$, $\nabla\rho$ blows up, so does $\nabla\partial_t\phi$.

Remark 4.8. *We compare the estimates (3.13) and (4.5). (3.13) is rough: $|\underline{L}\mu| \lesssim M^2$; (4.5) is precise: $|\underline{L}\mu| \leq C_0 + \delta M^2$, where C_0 depends only on the initial data. The improvement comes from integrating the wave equation $\square_{\widehat{g}}\psi_0 = 0$ or equivalently (4.1).*

4.2. The asymptotic expansion for derivatives of μ . We start with an estimate on derivatives of $\text{tr}\widetilde{\chi}$.

Lemma 4.9. *For sufficiently small $\delta \leq \varepsilon$, we have*

$$\|L\text{tr}_{\widetilde{g}}\widetilde{\chi}\|_{L^\infty} \lesssim M^2, \quad (4.8)$$

$$\|\not{d}\text{tr}\widetilde{\chi}\|_{L^\infty} \lesssim \delta M^2. \quad (4.9)$$

$$||t^2 \underline{L}(R_i \mu)(t, u, \theta) - |r_0|^2 \underline{L}(R_i \mu)(-r_0, u, \theta)|| \lesssim \delta M^4. \quad (4.14)$$

Since $\|[L, R_i]\|_{L^\infty} \lesssim \delta M^2$, we bound $R_i \mu$ as

$$\begin{aligned} R_i \mu(t, \underline{u}, \theta) - R_i \mu(-r_0, \underline{u}, \theta) &\stackrel{(4.14)}{=} \int_{-r_0}^t \frac{r_0^2 \underline{L}(R_i \mu(r_0, \underline{u}, \theta))}{\tau^2} d\tau + O(\delta M^4) \\ &= -\left(\frac{1}{t} + \frac{1}{r_0}\right) r_0^2 \underline{L}(R_i \mu(-r_0, \underline{u}, \theta)) + O(\delta M^4). \end{aligned}$$

This inequality yields (4.13) for sufficiently small δ . \square

We can also obtain a better estimate for $L\mu$.

Lemma 4.12. *For sufficiently small δ , we have*

$$\|L\mu\|_{L^\infty(\Sigma_t)} \lesssim \delta^{-1} + M^4. \quad (4.15)$$

Proof. By commuting L with $\underline{L}\mu = m + e\mu$, we obtain

$$\underline{L}(L\mu) = Lm + [-2(\underline{\zeta}^A + \underline{\eta}^A)X_A(\mu) + \underline{L}(c^{-2}\mu)\underline{L}\mu + eL\mu + \mu Le].$$

According to (B.1), we can bound the terms in the bracket by M^4 . Hence,

$$|t|^2 \underline{L}(L\mu)(t, \underline{u}, \theta) - |r_0|^2 \underline{L}(L\mu)(-r_0, \underline{u}, \theta) = |t|^2 Lm(t, \underline{u}, \theta) - |r_0|^2 Lm(-r_0, \underline{u}, \theta) + O(M^4).$$

By the explicit formula of m , we can proceed exactly as in Lemma 4.4 and we obtain

$$||t|^2 \underline{L}(L\mu)(t, \underline{u}, \theta) - |r_0|^2 \underline{L}(L\mu)(-r_0, \underline{u}, \theta)| \lesssim M^4. \quad (4.16)$$

We then integrate along \underline{L} and we have

$$\begin{aligned} L\mu(t, \underline{u}, \theta) - L\mu(-r_0, \underline{u}, \theta) &= \int_{-r_0}^t \frac{\tau^2 \underline{L}(L\mu(\tau, \underline{u}, \theta))}{\tau^2} d\tau \stackrel{(4.16)}{=} \int_{-r_0}^t \frac{r_0^2 \underline{L}(L\mu(r_0, \underline{u}, \theta))}{\tau^2} d\tau + \frac{O(M^4)}{\tau^2} d\tau \\ &= -\left(\frac{1}{t} + \frac{1}{r_0}\right) r_0^2 \underline{L}(L\mu(-r_0, \underline{u}, \theta)) + O(M^4). \end{aligned}$$

For sufficiently small δ , this implies (4.15). \square

We now relate $L^2\psi_0(t, \underline{u}, \theta)$ to its initial value.

Lemma 4.13. *For sufficiently small δ , we have*

$$|t|L^2\psi_0(t, \underline{u}, \theta) - r_0L^2\psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{-\frac{1}{2}}M^3. \quad (4.17)$$

Proof. We commute L with (4.1) and we obtain the following transport equation for $L^2\psi_0$:

$$\begin{aligned} \underline{L}(L^2\psi_0) + \frac{1}{2}\text{tr}\tilde{\chi} \cdot L^2\psi_0 &= -\frac{1}{2}L(\text{tr}\tilde{\chi}) \cdot L\psi_0 + [\underline{L}, L]L\psi_0 \\ &\quad + L\left(\mu\Delta\psi_0 - \frac{1}{2}\text{tr}\tilde{\chi} \cdot \underline{L}\psi_0 - 2\underline{\zeta} \cdot \not\Delta\psi_0 - \mu\not\Delta\log(c) \cdot \not\Delta\psi_0\right). \end{aligned} \quad (4.18)$$

The righthand side of the above equation can be expanded as

$$\begin{aligned} L\text{tr}\tilde{\chi}L\psi_0 + \underline{L}(c^{-2}\mu)\underline{L}L\psi_0 + (\underline{\eta} + \underline{\zeta})\not\Delta L\psi_0 + \left(L\mu\Delta\psi_0 + \mu L\Delta\psi_0 + L\text{tr}\tilde{\chi}\underline{L}\psi_0 + \text{tr}\tilde{\chi}\underline{L}L\psi_0\right. \\ \left.+ L\underline{\zeta}\not\Delta\psi_0 + \underline{\zeta}L\not\Delta\psi_0 + L\mu \cdot \not\Delta\log(c) \cdot \not\Delta\psi_0 + \mu L(\not\Delta\log(c)) \cdot \not\Delta\psi_0 + \mu\not\Delta\log(c) \cdot L\not\Delta\psi_0\right). \end{aligned}$$

Since the exact numeric constants and signs for the coefficients are irrelevant for estimates, we replace all of them by 1 in the above expressions.

Since $\underline{\zeta}_A = -c^{-1}\mu X_A(c)$, by applying L and using (4.15), we obtain $|L\underline{\zeta}| \lesssim M^2$. Therefore, according (4.8), (B.1) and the estimates derived previously in this section, we can bound all the terms on the right hand side and we obtain

$$|\underline{L}(L^2\psi_0) + \frac{1}{2}\text{tr}\tilde{\chi} \cdot L^2\psi_0| \lesssim \delta^{-\frac{1}{2}}M^3.$$

We then integrate from $-r_0$ to t to obtain (4.17). \square

We can commute L twice with (3.19) or with (4.1) to obtain following estimates.

Lemma 4.14. *For sufficiently small δ , we have*

$$\|L^2 \text{tr}_{\tilde{g}} \tilde{\chi}\| \lesssim M^2 \delta^{-1}, \quad (4.19)$$

$$|t| L^3 \psi_0(t, \underline{u}, \theta) - r_0 L^3 \psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{-\frac{3}{2}} M^3. \quad (4.20)$$

We omit the proof since it is routine. Similarly, we commute L twice with $\underline{L}\mu = m + e\mu$, we can use (4.19) and (4.20) to obtain

Lemma 4.15. *There exists $\varepsilon = \varepsilon(M)$ so that for all $\delta \leq \varepsilon$, we have*

$$\|L^2 \mu\|_{L^\infty(\Sigma_t)} \lesssim \delta^{-2} + \delta^{-1} M^2, \quad (4.21)$$

$$\|T^2 \mu\|_{L^\infty(\Sigma_t)} \lesssim \delta^{-2} + \delta^{-1} M^2. \quad (4.22)$$

We turn to another key property of μ which reflects the behavior of $\mu^{-1} T\mu$. (4.5) suggests that if shock forms before $t = -1$, $\underline{L}\mu$ to behave as a constant near s^* . Hence, μ is proportional to $|t - s^*|$. In view of (3.15), we expect $\mu^{-1} T\mu$ behaves as $|t - s^*|^{-1} \delta^{-1}$ near shocks (It is *not* integrable in t). The next proposition suggest a much better bound for $\mu^{-1} (T\mu)_+$ by affording one more derivative in T and we can improve $|t - s^*|^{-1}$ to $|t - s^*|^{-\frac{1}{2}}$.

Proposition 4.16. *For $p = (t, \underline{u}, \theta) \in W_\delta$, let $(\mu^{-1} T\mu)_+$ be the nonnegative part of $\mu^{-1} T\mu$. For sufficiently small δ and for all $p \in W_{shock}$, we have*

$$(\mu^{-1} T\mu)_+(t, \underline{u}, \theta) \lesssim \frac{1}{|t - s^*|^{\frac{1}{2}}} \delta^{-1}. \quad (4.23)$$

Proof. We use a maximal principle type argument. Let $\gamma : [0, \delta] \rightarrow W_\delta^*$ be the integral curve of T through the point p . We may choose the θ coordinates to be constant along γ on the given Σ_t and study the function $f(\underline{u}) = T(\log(\mu))(\underline{u})$. We assume $f(\underline{u})$ attains its maximum at a point $\underline{u}^* \in [0, \delta]$. We may also assume that this maximum is positive (Otherwise, (4.23) is automatically true). Since \underline{u}^* is a maximum point, we have $\frac{d}{d\underline{u}}(f)(\underline{u}^*) \geq 0$. In other words, $T(T(\log(\mu)))(\underline{u}^*) \geq 0$. Therefore, at the point $(t, \underline{u}^*, \theta)$, we have $\mu^{-1} T^2 \mu - \frac{1}{\mu^2} (T\mu)^2 \geq 0$. Hence,

$$\|\mu^{-1} (T\mu)_+\|_{L^\infty([0, \delta])} \leq \sqrt{\frac{\|T^2 \mu\|_{L^\infty([0, \delta])}}{\inf_{\underline{u} \in [0, \delta]} \mu(\underline{u})}}. \quad (4.24)$$

It suffices to bound the denominator and the numerator on the righthand side. For $T^2 \mu$, according to (4.22), we have

$$\|T^2 \mu\|_{L^\infty([0, \delta])} \lesssim \frac{1}{|t|} \delta^{-2} \lesssim \delta^{-2}. \quad (4.25)$$

For $\inf \mu$, we can assume $(t, \underline{u}, \theta) \in W_{shock}$. Otherwise we have the lower bound $\mu \geq \frac{1}{10}$. According to (4.7), the condition $(t, \underline{u}, \theta) \in W_{shock}$ implies $|t|^2 (\underline{L}\mu)(t, \underline{u}, \theta) \leq -\frac{1}{4}$. Therefore, by (4.5), we have

$$r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) = t^2 \underline{L}\mu(t, \underline{u}, \theta) + O(\delta M^2) \leq -\frac{1}{4} + O(\delta M^2) \leq -\frac{1}{6}$$

provided that δ is sufficiently small. We now integrate from t to s^* to derive

$$\begin{aligned}\mu(t, \underline{u}, \theta) &= \mu(s^*, \underline{u}, \theta) - \int_t^{s^*} \underline{L}\mu(\tau, \underline{u}, \theta) d\tau \geq - \int_t^{s^*} \underline{L}\mu(\tau, \underline{u}, \theta) d\tau = - \int_t^{s^*} \left(\frac{r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta)}{|\tau|^2} + O(\delta M^2) \right) d\tau \\ &\geq -r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \frac{1}{|s^*||t|} |t - s^*| + O(\delta M^2) |t - s^*|.\end{aligned}$$

Therefore, for sufficiently small δ , we obtain $|t - s^*| \lesssim \mu$. Together with (4.24) and (4.25), this completes the proof. \square

5. THE MECHANISM FOR SHOCK FORMATIONS

In this section, we assume Theorem 3.1 stated in Section 3.2 and we use knowledge on μ from last section to analyze a mechanism of shock formations. In particular, we have to use condition (1.6) in the **Main Theorem**. This is the only place in the paper where we use (1.6).

The transport equation $\underline{L}\mu = m + \mu e$ is responsible for the shock formations. We first give precise bounds on each term on the righthand side. Since $m = -\frac{1}{2} \frac{d}{d\rho} (c^2) T\rho$, we have $m = \frac{3G''(0)}{(1+3G''(0)\psi_0^2)^2} \psi_0 \cdot T\psi_0$.

In view of (4.3), up to an error of size $\delta^{\frac{1}{2}}$, we can replace $T\psi_0(t, \underline{u}, \theta)$ by $\frac{r_0}{|t|} T\psi_0(-r_0, \underline{u}, \theta)$; in view of (4.4), up to an error of size $\delta^{\frac{3}{2}}$, we can replace $\psi_0(t, \underline{u}, \theta)$ by $\frac{r_0}{|t|} \psi_0(-r_0, \underline{u}, \theta)$. Therefore, we obtain

$$m = 3G''(0) \frac{r_0^2}{|t|^2} \psi_0(-r_0, \underline{u}, \theta) T\psi_0(-r_0, \underline{u}, \theta) + O(\delta) = 3 \frac{r_0^2}{|t|^2} \left(G''(0) \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \partial_r \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \right) + O(\delta).$$

Since e is of size δ and $|\mu| \lesssim 1$, we regard μe as an error term. We then obtain

$$\underline{L}\mu(t, \underline{u}, \theta) = 3 \frac{r_0^2}{|t|^2} \left(G''(0) \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \partial_r \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \right) + O(\delta).$$

We then integrate this equation and we obtain

$$\begin{aligned}\mu(t, \underline{u}, \theta) - \mu(-r_0, \underline{u}, \theta) &= 3 \int_{-r_0}^t \frac{r_0^2}{|\tau|^2} d\tau \cdot \left(G''(0) \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \partial_r \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \right) + O(\delta) \\ &= 3r_0^2 \left(\frac{1}{|t|} - \frac{1}{r_0} \right) \left(G''(0) \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \partial_r \phi_1\left(\frac{r-r_0}{\delta}, \theta\right) \right) + O(\delta)\end{aligned}$$

Since $|\mu(-r_0, \underline{u}, \theta) - 1| \lesssim \delta$, according to condition (1.6), we have (recall that $r_0 = 2$)

$$\mu(t, \underline{u}, \theta) \leq 1 - 3 \cdot 2^2 \left(\frac{1}{|t|} - \frac{1}{2} \right) \frac{1}{6} + O(\delta) = 1 - 2 \left(\frac{1}{|t|} - \frac{1}{2} \right) + O(\delta)$$

Therefore, for sufficiently small δ , t can not be greater than -1 , otherwise μ would be negative. In other words, shock forms before $t = -1$.

Corollary 5.1. *If we introduce the vectorfield $\hat{T} := c\mu^{-1}T$, then when shock forms, the second derivative of ϕ , $\hat{T}^i \partial_i \partial_t \phi$, blows up.*

Proof. When shock forms, $\mu \rightarrow 0$, which means $\mu < \frac{1}{10}$. Then by (4.7), $\underline{L}\mu$ is negative and bounded from above. In other words, there is an absolute positive constant C such that $|\underline{L}\mu| \geq C$. While from the propagation equation $\underline{L}\mu = m + \mu e$ and the pointwise estimates $|e| \lesssim \delta, |\mu| \lesssim 1$, the estimate $|m| \geq C$ follows if δ is sufficiently small. By the definition $m = -\frac{1}{2} \frac{d(c^2)}{d\rho} T\rho$ and $\hat{T} = c\mu^{-1}T$, the derivative of ρ , $\hat{T}\rho = c\mu^{-1}T\rho$, blows up when $\mu \rightarrow 0$. Since $|\psi_0| \lesssim \delta^{1/2}$, $\hat{T}\psi_0 = \hat{T}^i \partial_i \partial_t \phi$ blows up when $\mu \rightarrow 0$ from the definition $\rho = \psi_0^2$. \square

Corollary 5.2. *When shock forms, the only nonzero curvature component in optical coordinates, α_{AB} , blows up.*

Proof. In the expression:

$$\underline{\alpha}_{AB} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_{X_A, X_B}^2 \rho - \frac{1}{2} \mu^{-1} \frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB} + \frac{1}{2} \left(\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \right) X_A(\rho) X_B(\rho),$$

the terms $\frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_{X_A, X_B}^2 \rho$, $\frac{1}{2} \left(\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \right) X_A(\rho) X_B(\rho)$, being derivatives of ρ in optical coordinates, are bounded pointwisely. While by (4.7) and (3.18), the pointwise norm of $\frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB}$ is bounded from below, therefore the term $\mu^{-1} \frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB}$ blows up when $\mu \rightarrow 0$. \square

The focus of the rest of the paper is to prove Theorem 3.1.

6. ENERGY ESTIMATES FOR LINEAR WAVE EQUATIONS

We study energy estimates for the linear wave equation

$$\square_{\tilde{g}} \psi = \rho \tag{6.1}$$

where ρ is a smooth function defined on W_δ . The energy momentum tensors for a solution ψ of (6.1) are the same for both g and \tilde{g} , i.e.,

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \psi \partial_\alpha \psi.$$

In the null frame $(X_1, X_2, L, \underline{L})$, $\tilde{T}_{\mu\nu}$ is decomposed as

$$\begin{aligned} T_{LL} &= (L\psi)^2, \quad T_{\underline{L}\underline{L}} = (\underline{L}\psi)^2, \quad T_{\underline{L}L} = \mu |\not{d}\psi|^2, \quad T_{LA} = L\psi \cdot X_A(\psi), \\ T_{\underline{L}A} &= \underline{L}\psi \cdot X_A(\psi), \quad T_{AB} = X_A(\psi) X_B(\psi) - \frac{1}{2} \not{g}_{AB} (-\mu^{-1} L\psi \underline{L}\psi + |\not{d}\psi|^2). \end{aligned} \tag{6.2}$$

We use two *multiplier vectorfields* $K_0 = L$ and $K_1 = \underline{L}$. The associated energy currents are defined by

$$\tilde{P}_0^\mu = -\tilde{T}^\mu{}_\nu K_0^\nu, \quad \tilde{P}_1^\mu = -\tilde{T}^\mu{}_\nu K_1^\nu. \tag{6.3}$$

The corresponding deformation tensors are denoted by $\tilde{\pi}_0 = {}^{(K_0)}\tilde{\pi}$ and $\tilde{\pi}_1 = {}^{(K_1)}\tilde{\pi}$ respectively. Since the divergence of $\tilde{T}_{\mu\nu}$ is $\tilde{\nabla}^\mu \tilde{T}_{\mu\nu} = \rho \cdot \partial_\nu \psi$, we have

$$\begin{aligned} \tilde{\nabla}_\mu \tilde{P}_0^\mu &= \tilde{Q}_0 := -\rho \cdot K_0 \psi - \frac{1}{2} \tilde{T}^{\mu\nu} \tilde{\pi}_{0,\mu\nu}, \\ \tilde{\nabla}_\mu \tilde{P}_1^\mu &= \tilde{Q}_1 := -\rho \cdot K_1 \psi - \frac{1}{2} \tilde{T}^{\mu\nu} \tilde{\pi}_{1,\mu\nu}. \end{aligned} \tag{6.4}$$

Let W_t^u be the spacetime region enclosed by $\Sigma_{-r_0}^u$, \underline{C}_0^t , Σ_t^u and \underline{C}_t^t . We integrate (6.4) on W_t^u under the condition that ψ and its derivatives vanish on \underline{C}_0^t (This is always the case in later applications), we obtain

$$\begin{aligned} E^0(t, \underline{u}) - E^0(-r_0, \underline{u}) + F^0(t, \underline{u}) &= \int_{W_t^u} c^{-2} \tilde{Q}_0, \\ E^1(t, \underline{u}) - E^1(-r_0, \underline{u}) + F^1(t, \underline{u}) &= \int_{W_t^u} c^{-2} \tilde{Q}_1. \end{aligned} \tag{6.5}$$

where the associated energy $E^i(t, \underline{u})$ and flux $F^i(t, \underline{u})$ are defined (naturally from the Stokes formula) as

$$\begin{aligned} E^0(t, \underline{u}) &= \int_{\Sigma_t^u} \frac{1}{2c} \left((L\psi)^2 + c^{-2} \mu^2 |\not{d}\psi|^2 \right), \quad F^0(t, \underline{u}) = \int_{\underline{C}_t^t} c^{-1} \mu |\not{d}\psi|^2, \\ E^1(t, \underline{u}) &= \int_{\Sigma_t^u} \frac{1}{2c} \left(c^{-2} \mu (\underline{L}\psi)^2 + \mu |\not{d}\psi|^2 \right), \quad F^1(t, \underline{u}) = \int_{\underline{C}_t^t} c^{-1} (\underline{L}\psi)^2. \end{aligned} \tag{6.6}$$

We emphasize that the integral on W_t^u defined as below on the spacetime slab contains a factor μ :

$$\int_{W_t^u} f = \int_{-r_0}^t \int_0^u \left(\int_{S_{\tau, \underline{u}'}} \mu \cdot f(\tau, \underline{u}', \theta) d\mu_g \right) d\underline{u}' d\tau.$$

We remark that, by (3.6) and (3.10), we have

$$\begin{aligned} E^0(t, \underline{u}) &\sim E(\psi)(t, \underline{u}), \quad F^0(t, \underline{u}) \sim F(\psi)(t, \underline{u}), \\ E^1(t, \underline{u}) &\sim \underline{E}(\psi)(t, \underline{u}), \quad F^1(t, \underline{u}) \sim \underline{F}(\psi)(t, \underline{u}). \end{aligned} \quad (6.7)$$

For the sake of simplicity, we use $E(t, \underline{u})$, $F(t, \underline{u})$, $\underline{E}(t, \underline{u})$ and $\underline{F}(t, \underline{u})$ as shorthand notations for $E(\psi)(t, \underline{u})$, $F(\psi)(t, \underline{u})$, $\underline{E}(\psi)(t, \underline{u})$ and $\underline{F}(\psi)(t, \underline{u})$ in the rest of the section.

We need to compute the so called *error integrals* or *error terms*, i.e., $\int_{W_t^u} \widetilde{Q}_0$ and $\int_{W_t^u} \widetilde{Q}_1$ in (6.5).

This requires an explicit formula for $\widetilde{\pi}_{0, \mu\nu}$ or $\widetilde{\pi}_{1, \mu\nu}$. The deformation tensor $\widetilde{\pi}_{0, \mu\nu}$ is given by

$$\begin{aligned} \widetilde{\pi}_{0, LL} &= 0, \quad \widetilde{\pi}_{0, L\underline{L}} = 0, \quad \widetilde{\pi}_{0, \underline{L}\underline{L}} = -2c^{-1}\mu \left(\mu^{-1}L\mu + L \log(c^{-1}) + \underline{L}(c^{-2}\mu) \right), \\ \widetilde{\pi}_{0, LA} &= -2c^{-1}\mu X_A(c^{-2}\mu), \quad \widetilde{\pi}_{0, \underline{L}A} = -2c^{-1}(\underline{\zeta}_A + \underline{\eta}_A), \quad \widetilde{\pi}_{0, AB} = 2\widetilde{\chi}_{AB}. \end{aligned} \quad (6.8)$$

The deformation tensor $\widetilde{\pi}_{1, \mu\nu}$ is given by

$$\begin{aligned} \widetilde{\pi}_{1, \underline{L}\underline{L}} &= 0, \quad \widetilde{\pi}_{1, LL} = 4c^{-1}\mu \underline{L}(c^{-2}\mu), \quad \widetilde{\pi}_{1, L\underline{L}} = -2c^{-1}\mu \left(\mu^{-1}\underline{L}\mu + \underline{L} \log(c^{-1}) \right), \\ \widetilde{\pi}_{1, \underline{L}A} &= 0, \quad \widetilde{\pi}_{1, LA} = 2c^{-1}(\underline{\zeta}_A + \underline{\eta}_A), \quad \widetilde{\pi}_{1, AB} = 2\underline{\chi}_{AB}. \end{aligned} \quad (6.9)$$

We also need an explicit formula for the energy momentum tensor $T^{\mu\nu}$:

$$\begin{aligned} T^{LL} &= \frac{(\underline{L}\psi)^2}{4\mu^2}, \quad T^{\underline{L}\underline{L}} = \frac{(L\psi)^2}{4\mu^2}, \quad T^{\underline{L}L} = \frac{(\not{L}\psi)^2}{4\mu}, \quad T^{LA} = -\frac{\underline{L}\psi X_A(\psi)}{2\mu}, \\ T^{\underline{L}A} &= -\frac{L\psi X_A(\psi)}{2\mu}, \quad T^{AB} = \not{g}^{AC} \not{g}^{BD} X_C(\psi) X_D(\psi) - \frac{1}{2} \not{g}^{AB} \left(-\frac{L\psi \underline{L}\psi}{\mu} + |\not{L}\psi|^2 \right). \end{aligned} \quad (6.10)$$

Finally, we can compute the integrands \widetilde{Q}_0 and \widetilde{Q}_1 explicitly. For \widetilde{Q}_0 , we have

$$\begin{aligned} c^{-2}\widetilde{Q}_0 &= -c^{-2}\rho \cdot K_0\psi - \frac{1}{2}T^{\mu\nu}\widetilde{\pi}_{0, \mu\nu} = Q_{0,0} + Q_{0,1} + Q_{0,2} + Q_{0,3} + Q_{0,4} \\ &= -c^{-2}\rho \cdot K_0\psi - T^{LL}\widetilde{\pi}_{0, LL} - T^{LA}\widetilde{\pi}_{0, LA} - T^{\underline{L}A}\widetilde{\pi}_{0, \underline{L}A} - \frac{1}{2}T^{AB}\widetilde{\pi}_{0, AB}. \end{aligned} \quad (6.11)$$

The $Q_{0,i}$'s are given explicitly by

$$\begin{aligned} Q_{0,1} &= \frac{1}{2c} \left(\mu^{-1}L\mu + L \log(c^{-1}) + \underline{L}(c^{-2}\mu) \right) |\not{L}\psi|^2, \\ Q_{0,2} &= -c^{-1}X_A(c^{-2}\mu)\underline{L}\psi \cdot X_A(\psi), \quad Q_{0,3} = -c^{-1}\mu^{-1}(\underline{\zeta}_A + \underline{\eta}_A)L\psi X_A(\psi), \\ Q_{0,4} &= -\frac{1}{2}(\widehat{\chi}_{AB}X^A\psi X^B\psi + \frac{c}{2}\mu^{-1}\text{tr}_{\widetilde{g}}\widetilde{\chi}L\psi \cdot \underline{L}\psi). \end{aligned} \quad (6.12)$$

For \widetilde{Q}_1 , we have

$$\begin{aligned} c^{-2}\widetilde{Q}_1 &= -c^{-2}\rho \cdot K_1\psi - \frac{1}{2}T^{\mu\nu}\widetilde{\pi}_{1, \mu\nu} = Q_{1,0} + Q_{1,1} + Q_{1,2} + Q_{1,3} + Q_{1,4} \\ &= -c^{-2}\rho \cdot K_1\psi - \frac{1}{2}T^{LL}\widetilde{\pi}_{1, LL} - T^{\underline{L}\underline{L}}\widetilde{\pi}_{1, \underline{L}\underline{L}} - T^{LA}\widetilde{\pi}_{1, LA} - \frac{1}{2}T^{AB}\widetilde{\pi}_{1, AB}. \end{aligned} \quad (6.13)$$

The $Q_{1,i}$'s are given explicitly by

$$\begin{aligned} Q_{1,1} &= -\frac{1}{2c\mu} \underline{L}(c^{-2}\mu) |\underline{L}\psi|^2, \quad Q_{1,2} = \frac{1}{2c} \left(\mu^{-1} \underline{L}\mu + \underline{L} \log(c^{-1}) \right) |\not{d}\psi|^2, \\ Q_{1,3} &= \frac{1}{c\mu} (\zeta_A + \eta_A) \underline{L}\psi \cdot X^A \psi, \quad Q_{1,4} = -\frac{1}{2} (\widehat{\chi}_{AB} X^A \psi X^B \psi + \frac{c}{2} \mu^{-1} \text{tr}_{\widetilde{g}} \widetilde{\chi} \underline{L}\psi \cdot \underline{L}\psi). \end{aligned} \quad (6.14)$$

6.1. Estimates on $Q_{1,2}$: the coercivity of energy estimates in shock region. We separate the principal terms and lower order terms of $Q_{1,2}$ as follows

$$Q_{1,2} = \frac{1}{2c} (\mu^{-1} \underline{L}\mu + \underline{L} \log(c^{-1})) |\not{d}\psi|^2 = (-\frac{1}{2c} \mu^{-1} \underline{L}\mu + \text{l.o.t.}) |\not{d}\psi|^2,$$

where the lower order terms l.o.t., thanks to (3.10), are bounded by $\|\text{l.o.t.}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2$. We rewrite the principal term $\int_{W_t^u} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2$ as

$$\int_{W_t^u} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 = \int_{W_t^u \cap W_{\text{shock}}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 + \int_{W_t^u \cap W_{\text{rare}}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2. \quad (6.15)$$

In view of the estimates (4.5) and the fact that $\mu \geq \frac{1}{10}$ in the rarefaction wave region W_{rare} , for sufficiently small δ , the second term is bounded by

$$\begin{aligned} \int_{W_t^u \cap W_{\text{rare}}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 &\lesssim \int_{W_t^u \cap W_{\text{rare}}} |\underline{L}\mu| |\not{d}\psi|^2 = \int_{W_t^u \cap W_{\text{rare}}} \left| \frac{|r_0|^2}{|t|^2} \underline{L}\mu(-r_0, \underline{u}, \theta) + \delta M^2 \right| |\not{d}\psi|^2 \\ &\lesssim \int_{W_t^u \cap W_{\text{rare}}} |\not{d}\psi|^2. \end{aligned}$$

Since $\int_{W_t^u \cap W_{\text{rare}}} |\not{d}\psi|^2 \leq \int_{W_t^u} |\not{d}\psi|^2 = \int_{-r_0}^t (\int_{\Sigma_\tau} \mu |\not{d}\psi|^2) d\tau$, we obtain

$$\int_{W_t^u \cap W_{\text{rare}}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 \lesssim \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau. \quad (6.16)$$

In the shock region, we make use of the key estimate (4.7). Therefore, we have

$$\int_{W_t^u \cap W_{\text{shock}}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 \leq - \int_{W_t^u \cap W_{\text{shock}}} \frac{1}{8c|t|} \mu^{-1} |\not{d}\psi|^2 \lesssim - \int_{W_t^u \cap W_{\text{shock}}} \mu^{-1} |\not{d}\psi|^2.$$

We define

$$K(t, \underline{u}) = \int_{W_t^u \cap W_{\text{shock}}} \mu^{-1} |\not{d}\psi|^2, \quad (6.17)$$

Therefore,

$$\int_{W_t^u \cap W_{\text{shock}}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 \lesssim -K(t, \underline{u}). \quad (6.18)$$

Finally, by combining (6.15), (6.16) and (6.18), for sufficiently small δ , we obtain

$$\int_{W_t^u} Q_{1,2} \lesssim -K(t, \underline{u}) + \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau. \quad (6.19)$$

The negative term $-K(t, \underline{u})$ in above estimates plays a key role to control $\not{d}\psi$ in error terms. It will compensate the degeneracy of the μ factor in front of $\not{d}\psi$ in the energy.

6.2. Estimates on $Q_{0,1}$. The estimates on $Q_{0,1}$ relies on the second key property (4.23) of μ . We first separate the principal terms and lower order terms of $Q_{0,1}$ as follows:

$$Q_{0,1} = \frac{1}{4c} \left(\mu^{-1} L \mu + L \log(c^{-1}) + \underline{L}(c^{-1} \kappa) \right) |\psi|^2 = \frac{1}{2c} \mu^{-1} T \mu |\psi|^2 + \text{l.o.t.},$$

where $\text{l.o.t.} = \frac{1}{4c} (c^{-2} \underline{L} \mu + L \log(c^{-1}) + \underline{L}(c^{-1} \kappa)) |\psi|^2$. According to the estimates derived in previous sections, the terms in the parentheses are bounded by M^2 . Hence,

$$\int_{W_t^u} \text{l.o.t.} \lesssim \int_{W_t^u} M^2 |\psi|^2 = M^2 \int_0^u \left(\int_{\underline{C}_{u'}} \mu |\psi|^2 \right) d\underline{u}' \leq M^2 \int_0^u F(t, \underline{u}') d\underline{u}'. \quad (6.20)$$

For the principal term whose integrand is $\frac{1}{2c} \mu^{-1} T \mu |\psi|^2 \sim \mu^{-1} T \mu |\psi|^2$, due to the positivity of $|\psi|^2$, one ignores the contribution from the negative part $\frac{1}{c} (\mu^{-1} T \mu)_- |\psi|^2$. Therefore, it is bounded by

$$\lesssim \int_{W_t^u} (\mu^{-1} T \mu)_+ |\psi|^2 = \int_{W_t^u \cap W_{rare}} (\mu^{-1} T \mu)_+ |\psi|^2 + \int_{W_t^u \cap W_{shock}} (\mu^{-1} T \mu)_+ |\psi|^2.$$

In W_{rare} , since $\mu^{-1} \lesssim 1$, we have $\int_{W_t^u \cap W_{rare}} (\mu^{-1} T \mu)_+ |\psi|^2 \lesssim \int_{W_t^u \cap W_{rare}} |T \mu| |\psi|^2 \leq \int_{W_t^u} |T \mu| |\psi|^2$. According to (4.15), we have $T \mu \lesssim \delta^{-1}$, therefore,

$$\int_{W_t^u \cap W_{rare}} (\mu^{-1} T \mu)_+ |\psi|^2 \lesssim \int_{W_t^u} \delta^{-1} |\psi|^2 \leq \delta^{-1} \int_0^u F(t, \underline{u}') d\underline{u}'. \quad (6.21)$$

In W_{shock} , the argument relies on Proposition 4.16. Indeed, we have

$$\int_{W_t^u \cap W_{shock}} (\mu^{-1} T \mu)_+ |\psi|^2 \lesssim \int_{W_t^u \cap W_{shock}} \frac{1}{|t - s^*|^{\frac{1}{2}}} \delta^{-1} |\psi|^2 \leq \delta^{-1} \int_{W_t^u} \frac{1}{|t - t^*|^{\frac{1}{2}}} |\psi|^2.$$

By definition, the last integral is equal to $\delta^{-1} \int_{-r_0}^t \frac{1}{|\tau - t^*|^{\frac{1}{2}}} \left(\int_{\Sigma_\tau} \mu |\psi|^2 d\mu_\sharp \right) d\tau$. In view of the definition of $\underline{E}(t, \underline{u})$, we obtain

$$\int_{W_t^u \cap W_{shock}} (\mu^{-1} T \mu)_+ |\psi|^2 \lesssim \delta^{-1} \int_{-r_0}^t \frac{1}{|\tau - t^*|^{\frac{1}{2}}} \underline{E}(\tau, \underline{u}) d\tau \quad (6.22)$$

We remark that the key feature of above estimates is that the factor $\frac{1}{|t - t^*|^{\frac{1}{2}}}$ is integrable in t . It will allow us to use Gronwall's inequality.

Finally, taking into account of the estimates (6.20), (6.21) and (6.22), for sufficiently small δ , we obtain

$$\int_{W_t^u} Q_{0,1} \lesssim \delta^{-1} \int_{-r_0}^t \frac{1}{|\tau - t^*|^{\frac{1}{2}}} \underline{E}(\tau, \underline{u}) d\tau + \delta^{-1} \int_0^u F(t, \underline{u}') d\underline{u}'. \quad (6.23)$$

6.3. Estimates on other error terms. We deal with $Q_{0,2}$, $Q_{0,3}$, $Q_{0,4}$, $Q_{1,1}$, $Q_{1,3}$ and $Q_{1,4}$ one by one.

For $Q_{0,2}$, we have $|Q_{0,2}| = | -\frac{1}{c} X^A (c^{-2} \mu) \underline{L} \psi \cdot X_A(\psi) | \lesssim |\psi| |\underline{L} \psi| |\psi|$. According to (4.13), $|\psi| \lesssim 1 + \delta M^4$. Hence, for sufficiently small δ , we have

$$| \int_{W_t^u} Q_{0,2} | \lesssim \int_{W_t^u} (1 + \delta M^4) |\underline{L} \psi| |\psi| \lesssim \int_{W_t^u} |\underline{L} \psi|^2 + |\psi|^2$$

Therefore, we obtain

$$| \int_{W_t^u} Q_{0,2} | \lesssim \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau. \quad (6.24)$$

For $Q_{0,3}$, we first recall that $|\underline{\zeta}| \lesssim 1$ and $\underline{\eta} = \underline{\zeta} + \underline{\psi}\mu$, therefore, $|\underline{\zeta}| + |\underline{\eta}| \lesssim 1 + \delta M^4$. We break $Q_{0,3}$ into two parts as follows:

$$\begin{aligned} \left| \int_{W_t^{\underline{u}}} Q_{0,3} \right| &\lesssim \int_{W_t^{\underline{u}}} \mu^{-1} (|\underline{\zeta}| + |\underline{\eta}|) |L\psi| |\underline{\psi}| \lesssim \int_{W_t^{\underline{u}}} \mu^{-1} |L\psi| |\underline{\psi}| = I_1 + I_2 \\ &= \int_{W_t^{\underline{u}} \cap W_{rare}} \mu^{-1} |L\psi| |\underline{\psi}| + \int_{W_t^{\underline{u}} \cap W_{shock}} \mu^{-1} |L\psi| |\underline{\psi}|. \end{aligned}$$

The integral $I_1 = \int_{W_t^{\underline{u}} \cap W_{rare}} \mu^{-1} |L\psi| |\underline{\psi}|$ is taken in W_{rare} where $\mu \geq \frac{1}{10}$, therefore, we obtain

$$\begin{aligned} I_1 &\leq \int_{W_t^{\underline{u}} \cap W_{rare}} \mu^{-1} (|L\psi|^2 + |\underline{\psi}|^2) \lesssim \int_{W_t^{\underline{u}} \cap W_{rare}} \mu^{-1} (|L\psi|^2 + \mu |\underline{\psi}|^2) \lesssim \int_{W_t^{\underline{u}}} \mu^{-1} (|L\psi|^2 + \mu |\underline{\psi}|^2) \\ &\lesssim \int_{-r_0}^t E(\tau, \underline{u}) d\tau + \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}'. \end{aligned}$$

To control the integral $I_2 = \int_{W_t^{\underline{u}} \cap W_{shock}} \mu^{-1} |L\psi| |\underline{\psi}|$, we use the coercive term $K(t, \underline{u})$ from $Q_{1,2}$ to control the loss of μ^{-1} :

$$I_2 \leq \left(\int_{W_t^{\underline{u}} \cap W_{shock}} \mu^{-1} |\underline{\psi}|^2 \right)^{\frac{1}{2}} \left(\int_{W_t^{\underline{u}} \cap W_{shock}} \mu^{-1} |L\psi|^2 \right)^{\frac{1}{2}} \lesssim K(t, \underline{u})^{\frac{1}{2}} \left(\int_{-r_0}^t E(\tau, \underline{u}) d\tau \right)^{\frac{1}{2}}.$$

Hence, we obtain

$$\left| \int_{W_t^{\underline{u}}} Q_{0,3} \right| \lesssim \int_{-r_0}^t E(\tau, \underline{u}) d\tau + \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}' + K(t, \underline{u})^{\frac{1}{2}} \left(\int_{-r_0}^t E(\tau, \underline{u}) d\tau \right)^{\frac{1}{2}}. \quad (6.25)$$

For $Q_{0,4}$, we have

$$\left| \int_{W_t^{\underline{u}}} Q_{0,4} \right| \lesssim \int_{W_t^{\underline{u}}} |\widehat{\chi}| |\underline{\psi}|^2 + \mu^{-1} |\text{tr}_g \widehat{\chi}| |L\psi| |\underline{\psi}| \lesssim \int_{W_t^{\underline{u}}} \delta M^2 |\underline{\psi}|^2 + \mu^{-1} |L\psi| |\underline{\psi}| = I_1 + I_2.$$

The bound on I_1 is immediate: $I_1 = \delta M^2 \int_{W_t^{\underline{u}}} |\underline{\psi}|^2 \leq \delta M^2 \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}'$. The bound on $I_2 = \int_{W_t^{\underline{u}}} \mu^{-1} |L\psi| |\underline{\psi}|$ relies on the energy $E(t, \underline{u})$ and on the flux $\underline{F}(t, \underline{u})$:

$$I_2 \lesssim \int_{W_t^{\underline{u}}} \mu^{-1} |L\psi|^2 + \int_{W_t^{\underline{u}}} \mu^{-1} |\underline{\psi}|^2 = \int_{-r_0}^t E(\tau, \underline{u}) d\tau + \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}'.$$

Hence, we obtain

$$\left| \int_{W_t^{\underline{u}}} Q_{0,4} \right| \lesssim \int_{-r_0}^t E(\tau, \underline{u}) d\tau + \delta M^2 \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}' + \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}'. \quad (6.26)$$

For $Q_{1,1}$, according to (4.5), we have $|Q_{1,1}| = \left| \frac{1}{2c\mu} \underline{L}(c^{-2}\mu) |\underline{\psi}|^2 \right| \lesssim |\mu^{-1} \underline{L}\mu |\underline{\psi}|^2| \lesssim \mu^{-1} |\underline{\psi}|^2$. Hence,

$$\left| \int_{W_t^{\underline{u}}} Q_{1,1} \right| \lesssim \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}'. \quad (6.27)$$

For $Q_{1,3}$, since $|\underline{\zeta}| + |\underline{\eta}| \lesssim 1$, we break the integral into two parts:

$$\begin{aligned} \left| \int_{W_t^{\underline{u}}} Q_{1,3} \right| &\lesssim \int_{W_t^{\underline{u}}} \mu^{-1} |\underline{\psi}| |\underline{\psi}| = I_1 + I_2 \\ &= \int_{W_t^{\underline{u}} \cap W_{rare}} \mu^{-1} |\underline{\psi}| |\underline{\psi}| + \int_{W_t^{\underline{u}} \cap W_{shock}} \mu^{-1} |\underline{\psi}| |\underline{\psi}|. \end{aligned}$$

In the rarefaction wave region, since $\mu^{-1} \sim 1$, we obtain

$$\begin{aligned} I_1 &\lesssim \int_{W_t^u \cap W_{rare}} \mu^{-1} (|\underline{L}\psi|^2 + |\underline{d}\psi|^2) \approx \int_{W_t^u \cap W_{rare}} |\underline{L}\psi|^2 + |\underline{d}\psi|^2 \\ &\lesssim \int_{W_t^u} |\underline{L}\psi|^2 + |\underline{d}\psi|^2 \lesssim \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau. \end{aligned}$$

In the shock wave region, we still use $K(t, \underline{u})$ to control the loss of μ^{-1} :

$$I_2 \leq \left(\int_{W_t^u \cap W_{shock}} \mu^{-1} |\underline{d}\psi|^2 \right)^{\frac{1}{2}} \left(\int_{W_t^u \cap W_{shock}} \mu^{-1} |\underline{L}\psi|^2 \right)^{\frac{1}{2}} \lesssim K(t, \underline{u})^{\frac{1}{2}} \left(\int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}' \right)^{\frac{1}{2}}.$$

Hence,

$$\left| \int_{W_t^u} Q_{1,3} \right| \lesssim \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau + K(t, \underline{u})^{\frac{1}{2}} \left(\int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}' \right)^{\frac{1}{2}}. \quad (6.28)$$

For $Q_{1,4}$, we have

$$\begin{aligned} \left| \int_{W_t^u} Q_{1,4} \right| &\lesssim \int_{W_t^u} |\widehat{\underline{\chi}}| |\underline{d}\psi|^2 + \mu^{-1} |\text{tr}_{\widehat{g}} \widehat{\underline{\chi}}| |\underline{L}\psi| |\underline{L}\psi| \lesssim \int_{W_t^u} \delta M^2 |\underline{d}\psi|^2 + \mu^{-1} |\underline{L}\psi| |\underline{L}\psi| \\ &\lesssim \delta M^2 \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}' + \left(\delta \int_{W_t^u} \mu^{-1} |\underline{L}\psi|^2 + \delta^{-1} \int_{W_t^u} \mu^{-1} |\underline{L}\psi|^2 \right). \end{aligned}$$

Therefore, we obtain

$$\left| \int_{W_t^u} Q_{1,4} \right| \lesssim \delta \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau + \delta M^2 \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}' + \delta^{-1} \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}'. \quad (6.29)$$

6.4. Summary. In view of (6.5) and (6.7), since ψ vanishes to infinite order on \underline{C}_0 , we have

$$\begin{aligned} \left(E(t, \underline{u}) + F(t, \underline{u}) \right) + \delta^{-1} \left(\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) \right) &\lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) \\ &\quad + \sum_{i=1}^4 \int_{W_t^u} Q_{0,i} + \delta^{-1} \sum_{i=1}^4 \int_{W_t^u} Q_{1,i} + \left| \int_{W_t^u} \rho \cdot \underline{L}\psi \right| + \delta^{-1} \left| \int_{W_t^u} \rho \cdot \underline{L}\psi \right|. \end{aligned}$$

We bound sums $\sum_{i=1}^4 \int_{W_t^u} Q_{0,i}$ and $\sum_{i=1}^4 \int_{W_t^u} Q_{1,i}$ by (6.19), (6.23), (6.24), (6.25), (6.26), (6.27), (6.28) and (6.29). Therefore, we have

$$\begin{aligned} &\left(E(t, \underline{u}) + F(t, \underline{u}) \right) + \delta^{-1} \left(\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) \right) \\ &\lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) + \left| \int_{W_t^u} \rho \cdot \underline{L}\psi \right| + \delta^{-1} \left| \int_{W_t^u} \rho \cdot \underline{L}\psi \right| \\ &\quad + \underbrace{\delta^{-1} \left(-K(t, \underline{u}) + \int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau \right)}_{Q_{1,2}} + \underbrace{\left(\delta^{-1} \int_{-r_0}^t \frac{1}{|\tau - t^*|^{\frac{1}{2}}} \underline{E}(\tau, \underline{u}) d\tau + \delta^{-1} \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}' \right)}_{Q_{0,1}} \\ &\quad + \underbrace{\left(\int_{-r_0}^t [E(\tau, \underline{u}) + \underline{E}(\tau, \underline{u})] d\tau + \int_0^{\underline{u}} [F(t, \underline{u}') + \underline{F}(t, \underline{u}')] d\underline{u}' + K(t, \underline{u})^{\frac{1}{2}} \left(\int_{-r_0}^t \underline{E}(\tau, \underline{u}) d\tau \right)^{\frac{1}{2}} \right)}_{Q_{0,2} + Q_{0,3} + Q_{0,4}} \\ &\quad + \delta^{-1} \underbrace{\left(\int_{-r_0}^t [\delta E(\tau, \underline{u}) + \underline{E}(\tau, \underline{u})] d\tau + \int_0^{\underline{u}} [\delta M^2 F(t, \underline{u}') + \delta^{-1} \underline{F}(t, \underline{u}')] d\underline{u}' + K(t, \underline{u})^{\frac{1}{2}} \left(\int_{-r_0}^t \underline{F}(t, \underline{u}') d\underline{u}' \right)^{\frac{1}{2}} \right)}_{Q_{1,1} + Q_{1,3} + Q_{1,4}}. \end{aligned}$$

For sufficiently small δ , we can rewrite the estimates as

$$\begin{aligned}
& \left(E(t, \underline{u}) + F(t, \underline{u}) \right) + \delta^{-1} \left(\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) \right) \\
& \lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) + \left| \int_{W_t^{\underline{u}}} \rho \cdot L\psi \right| + \delta^{-1} \left| \int_{W_t^{\underline{u}}} \rho \cdot \underline{L}\psi \right| \\
& \quad - \delta^{-1} K(t, \underline{u}) + \delta^{-1} \int_{-r_0}^t \frac{1}{|\tau - t^*|^{\frac{1}{2}}} \underline{E}(\tau, \underline{u}) d\tau \\
& \quad + \underbrace{\delta^{-1} K(t, \underline{u})^{\frac{1}{2}} \left(\delta \left(\int_{-r_0}^t E(\tau, \underline{u}) d\tau \right)^{\frac{1}{2}} + \left(\int_0^t \underline{F}(t, \underline{u}') d\underline{u}' \right)^{\frac{1}{2}} \right)}_I \\
& \quad + \left[\int_{-r_0}^t [E(\tau, \underline{u}) + \delta^{-1} \underline{E}(\tau, \underline{u})] d\tau + \delta^{-1} \int_0^{\underline{u}} [F(t, \underline{u}') + \delta^{-1} \underline{F}(t, \underline{u}')] d\underline{u}' \right]
\end{aligned}$$

For the term involving I , we use Cauchy-Schwarz inequality and put a small parameter ε_0 in front of the $K(t, \underline{u})$, i.e, $I \lesssim \delta^{-1} \varepsilon_0 K(t, \underline{u}) + \delta^{-1} \frac{1}{\varepsilon_0} \left(\delta \int_{-r_0}^t E(\tau, \underline{u}) d\tau + \int_{-r_0}^t \underline{F}(t, \underline{u}') d\underline{u}' \right)$. Therefore, the resulting $K(t, \underline{u})$ term can be absorbed by the coercive term $-K(t, \underline{u})$ and we obtain

$$\begin{aligned}
& \left(E(t, \underline{u}) + F(t, \underline{u}) \right) + \delta^{-1} \left(\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) \right) \\
& \lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) + \left| \int_{W_t^{\underline{u}}} \rho \cdot L\psi \right| + \delta^{-1} \left| \int_{W_t^{\underline{u}}} \rho \cdot \underline{L}\psi \right| \\
& \quad - \delta^{-1} K(t, \underline{u}) + \delta^{-1} \int_{-r_0}^t \frac{1}{|\tau - t^*|^{\frac{1}{2}}} \underline{E}(\tau, \underline{u}) d\tau \\
& \quad + \left[\int_{-r_0}^t E(\tau, \underline{u}) + \delta^{-1} \underline{E}(\tau, \underline{u}) d\tau + \delta^{-1} \int_0^{\underline{u}} F(t, \underline{u}') + \delta^{-1} \underline{F}(t, \underline{u}') d\underline{u}' \right]
\end{aligned}$$

By Gronwall's inequality (the factor $\frac{1}{|\tau - t^*|^{\frac{1}{2}}}$ is integrable in τ !), we can remove all the integral terms on the last line, this proves the Fundamental Energy Estimates **(F.E.E)** for $\square_{\tilde{g}}\psi = \rho$:

$$\begin{aligned}
& \left(E(t, \underline{u}) + F(t, \underline{u}) \right) + \delta^{-1} \left(\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) + K(t, \underline{u}) \right) \\
& \lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) + \left| \int_{W_t^{\underline{u}}} \rho \cdot L\psi \right| + \delta^{-1} \left| \int_{W_t^{\underline{u}}} \rho \cdot \underline{L}\psi \right|. \tag{F.E.E}
\end{aligned}$$

Remark 6.1. We see that the energy estimates only imply

$$\underline{E}(t, \underline{u}) + K(t, \underline{u}) \lesssim \delta,$$

which does not recover the full regularity of $\underline{L}\psi$ and ψ with respect to δ , as indicated by bootstrap assumption. We will see finally that this full regularity in δ is recovered by using the estimates for $E(t, \underline{u})$ together with Lemma 7.3 and the commutation of Q .

7. COMPARISONS BETWEEN EUCLIDEAN AND OPTICAL GEOMETRIES

There are two different geometries coming into play on W_δ^* , namely, the Minkowski geometry and the optical geometry. The knowledge on two metrics $g_{\mu\nu}$ (or $\tilde{g}_{\mu\nu}$) and $m_{\mu\nu}$ is essentially tied to the estimates on the solution of (\star) . There are many ways to compare two geometries, e.g., we may consider the Cartesian coordinates x^k as functions of the optical coordinates $(t, \underline{u}, \theta)$. In what follows, we also study other quantities as y^k, z^k, λ_i , etc. As a by product, we will also obtain estimates for the lower order objects, i.e., with order $< N_{\text{top}} + 1$.

Given a vectorfield V , we define the null components of its deformation tensor as

$${}^{(V)}\underline{Z}_A = {}^{(V)}\pi(\underline{L}, X_A), \quad {}^{(V)}Z_A = {}^{(V)}\pi(L, X_A), \quad {}^{(V)}\not{A}_{AB} = {}^{(V)}\pi(X_A, X_B). \quad (7.1)$$

The projection of Lie derivative \mathcal{L}_V to $S_{t,\underline{u}}$ is denoted as \mathcal{L}_V . The shorthand notation $\mathcal{L}_{Z_i}^\alpha$ to denote $\mathcal{L}_{Z_{i_1}} \mathcal{L}_{Z_{i_2}} \cdots \mathcal{L}_{Z_{i_k}}$ for a multi-index $\alpha = (i_1, \dots, i_k)$. We will show that, for all $|\alpha| \leq N_\infty$, we have $\mathcal{L}_{Z_i}^{\alpha-1} \underline{\chi}', \mathcal{L}_{Z_i}^{\alpha-1(Z_j)} \underline{Z}, \mathcal{L}_{Z_i}^{\alpha-1(Z_j)} \not{A}, \mathcal{L}_{Z_i}^{\alpha-1(Q)} (\not{A} + 4) \in \mathcal{O}_{2-2l}^{|\alpha|}$, where l is the number of T 's in Z_i 's, $|\alpha| \geq 1$ and $Z_j \neq T$. If $Z_j = T$, then we have $\mathcal{L}_{Z_i}^{\alpha-1(T)} \underline{Z}, \mathcal{L}_{Z_i}^{\alpha-1(T)} \not{A} \in \mathcal{O}_{-2l}^{|\alpha|}$. The idea of the proof is to compare $g_{\mu\nu}$ and $m_{\mu\nu}$ via quantities such as x^i, y^i, z^i, \hat{T}^i and λ_i . Similarly, we will derive L^2 -estimates on objects of order $\leq N_\mu$. The L^2 estimates depend on the L^∞ estimates up to order $N_\infty + 2$. In the course of the proof, it will be clear why N_∞ is chosen to be approximately $\frac{1}{2}N_{top}$.

7.1. L^∞ estimates. We assume (B.1): for all $|\alpha| \leq N_\infty$, $Z_i^{\alpha+2}\psi \in \Psi_{1-2l}^{|\alpha|+2}$.

Proposition 7.1. *For sufficiently small δ , for all $|\alpha| \leq N_\infty$ and $t \in [-r_0, s^*]$, we can bound $\{\mathcal{L}_{Z_i}^\alpha \underline{\chi}', \mathcal{L}_{Z_i}^{\alpha(Z_j)} \underline{Z}, \mathcal{L}_{Z_i}^{\alpha(Z_j)} \not{A}, \mathcal{L}_{Z_i}^{\alpha(Q)} (\not{A} + 4), Z_i^{\alpha+1} y^j, Z_i^{\alpha+1} \lambda_j\} \subset \mathcal{O}_{2-2l}^{|\alpha|+1}$ and $\{Z_i^{\alpha+1} x^j, Z_i^{\alpha+1} \hat{T}^j, \mathcal{L}_{Z_i}^{\alpha+1} Z_j, \mathcal{L}_{Z_i}^{\alpha(T)} \underline{Z}, \mathcal{L}_{Z_i}^{\alpha(T)} \not{A}\} \subset \mathcal{O}_{-2l}^{|\alpha|+1}$ in terms of $Z_i^{\alpha+2}\psi \in \Psi_{1-2l}^{|\alpha|+2}$. Here l is the number of T 's in Z_i 's.*

The estimates on x^j should be viewed as a good comparison between Euclidean and optical geometries on each slice Σ_t .

Proof. We do induction on the order. When $|\alpha| = 0$, the estimates are treated in Section 3. Here we only treat the estimates when $l = 0$, when $l \geq 1$, we can use the structure equation (2.34) to reduce the problem to the estimates for μ , which will be treated in Proposition 7.2. Here the loss of δ in the estimates for $\mathcal{L}_{Z_i}^{\alpha(T)} \underline{Z}, \mathcal{L}_{Z_i}^{\alpha(T)} \not{A}$ comes from applying T to $\frac{2}{t-\underline{u}}$, which is the principal part of $\text{tr} \underline{\chi}$. Given $|\alpha| \leq N_\infty$,

we assume that estimates hold for terms of order $\leq |\alpha|$. In particular, we have $\mathcal{L}_{R_i}^\beta \underline{\chi}', R_i^{\beta+1} y^j \in \mathcal{O}_2^{|\beta|}$ for all $|\beta| \leq |\alpha|$. We prove the proposition for $|\alpha| + 1$.

Step 1 Bounds $R_i^{\alpha+1} x^j$. Let $\delta_{\alpha+1,i}^j = \Omega_i^{\alpha+1} x^j - R_i^{\alpha+1} x^j$ where Ω_i 's are the standard rotational vectorfields on Euclidean space. It is obvious that $\Omega_i^{\alpha+1} x^j$ is equal to some x^k , therefore, bounded by r hence by a universal constant. Since $R_i = \Omega_i - \lambda_i \hat{T}^j \partial_j$, $R_i x^j \in \mathcal{O}_0^0$ and by ignoring all the numerical constants, we have

$$\delta_{\alpha+1,i}^j = R_i^\alpha (\lambda_i \hat{T}^j) = R_i^\alpha \left(\lambda_i \left(\frac{x^j}{\underline{u}-t} + y^j \right) \right).$$

Here the index i is not a single index. It means we apply a string of different R_i 's. This notation applies in the following when a string of R_i 's are considered.

Since the above expression has total order $\leq |\alpha|$, by the induction hypothesis we obtain immediately that $\delta_{\alpha+1,i}^j \in \mathcal{O}_2^{|\alpha|}$. In view of the definition of $\delta_{\alpha+1,i}^j$, we then have $R_i^{\alpha+1} x^j \in \mathcal{O}_0^{|\alpha|+1}$.

Step 2 Bounds on $\mathcal{L}_{R_i}^\alpha \underline{\chi}'$. We commute $\mathcal{L}_{R_i}^\alpha$ with (3.19) to derive

$$\begin{aligned} \mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}' &= [\mathcal{L}_{\underline{L}}, \mathcal{L}_{R_i}^\alpha] \underline{\chi}' + e \cdot \mathcal{L}_{R_i}^\alpha \underline{\chi}' + \underline{\chi}' \cdot \mathcal{L}_{R_i}^\alpha \underline{\chi}' + \sum_{\substack{|\beta_1|+|\beta_2|=|\alpha| \\ |\beta_2|<|\alpha|}} R_i^{\beta_1} e \cdot \mathcal{L}_{R_i}^{\beta_2} \underline{\chi}' \\ &+ \sum_{\substack{|\beta_1|+|\beta_2|+|\beta_3|=|\alpha| \\ |\beta_1|<|\alpha|, |\beta_3|<|\alpha|}} \mathcal{L}_{R_i}^{\beta_1} \underline{\chi}' \cdot \mathcal{L}_{R_i}^{\beta_2} \not{A}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} \underline{\chi}' + \mathcal{L}_{R_i}^\alpha \left(\frac{e \not{A}_{AB}}{t-\underline{u}} - \alpha'_{AB} \right). \end{aligned} \quad (7.2)$$

$$\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}' = [\mathcal{L}_{\underline{L}}, \mathcal{L}_{R_i}^\alpha] \underline{\chi}' + \mathcal{O}_2^1 \cdot \mathcal{L}_{R_i}^{\alpha-1} \underline{\chi}' + \Psi_2^{\leq |\alpha|+2}.$$

$$\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \chi' = \mathcal{O}_2^1 \cdot \mathcal{L}_{R_i}^\alpha \chi' + \Psi_2^{\leq |\alpha|+2}.$$

Step 3 Bounds on $R_i^{\alpha+1}y^j$ and $R_i^{\alpha+1}\lambda_j$. Since $R_i y^j = (-c^{-1}\underline{\chi}_B^A - \frac{\delta_B^A}{\underline{u}-t})R_i^A \not{d}_B x^j$, schematically we have

$$R_i^{\alpha+1}y^j = R_i^\alpha R_k y^j = R_i^\alpha \left(\left(c^{-1}\underline{\chi} + \frac{\delta}{\underline{u}-t} \right) \cdot R_k \cdot dx^j \right).$$

We distribute R_i^α inside the parenthesis by Leibniz rule. (Here again, the index i is not a single index, so we use index k to distinct the last rotation vectorfield.) Therefore, a typical term would be either $\mathcal{L}_{R_i}^{\beta_1} \chi \cdot \mathcal{L}_{R_i}^{\beta_2} g^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_k \lrcorner R_i^{\beta_4} x^j$ or $\mathcal{L}_{R_i}^{\beta_1} g \cdot \mathcal{L}_{R_i}^{\beta_2} g^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_k \lrcorner R_i^{\beta_4} x^j$ with $|\beta_1| + |\beta_2| + |\beta_3| + |\beta_4| = |\alpha|$.

There are only two terms where are not included in the induction hypothesis: $\mathcal{L}_{R_i}^{\beta_2} \mathcal{J}_{AB}$ and $\mathcal{L}_{R_i}^{\beta_3} R_j$. The first term is in fact easy to handle by induction hypothesis and estimates derived in Step 1 and Step 2, since $\mathcal{L}_{R_i} \mathcal{J}_{AB} = {}^{(R_i)}\!\!\not\!\mathcal{A}_{AB} = 2\lambda_i c^{-1} \underline{\chi}_{AB}$. For the second one, we use the following expression:

$$\begin{aligned} \mathcal{L}_{R_i} R_j = & - \sum_{k=1}^3 \varepsilon_{ijk} R_k + \lambda_i \left(\frac{c^{-1} - 1}{\underline{u} - t} \not{g} - c^{-1} (\underline{\chi} + \frac{\not{g}}{\underline{u} - t}) \right) R_j - \lambda_j \left(\frac{c^{-1} - 1}{\underline{u} - t} \not{g} - c^{-1} (\underline{\chi} + \frac{\not{g}}{\underline{u} - t}) \right) R_i \\ & - \lambda_i \varepsilon_{jkl} y^k \not{d} x^l \cdot \not{g}^{-1} + \lambda_j \varepsilon_{ikl} y^k \not{d} x^l \cdot \not{g}^{-1}. \end{aligned}$$

Therefore, $\mathcal{L}_{R_i}^{\beta_3} R_k = \mathcal{L}_{R_i}^{\beta_3-1} \mathcal{L}_{R_i} R_k = \mathcal{O}_0^{|\beta_3|-1} + \mathcal{O}_{\geq 2}^{|\beta_3|-1} R_i^{\beta_3} x^j$. Finally, we obtain that

$$R_i^{\alpha+1}y^j = \mathcal{O}_2^{\leq|\alpha|} + \mathcal{O}_2^{\leq|\alpha|} \cdot dR_i^{\beta_4}x^j.$$

Although $\#R_i^{\beta_4}x^j$ and $\mathcal{L}_{R_i}^{\beta_1}\chi$ may have order $|\alpha| + 1$, they have been controlled from previous steps. This gives the bounds on $R_i^{\alpha+1}y^j$. Then by the fact that $\lambda_j = \epsilon_{jkl}x^ky^l$, the estimate for $R_i^{\alpha+1}\lambda_j$ follows. This completes the proof of the proposition. \square

Proposition 7.2. *For sufficiently small δ , for all $|\alpha| \leq N_\infty, t \in [-r_0, s^*]$, we can bound $Z_i^{\alpha+1} \mu \in \mathcal{O}_{-2l}^{|\alpha|+1}$ in terms of $Z_i^{\alpha+2} \psi \in \Psi_{1-2l}^{|\alpha|+2}$.*

Proof. We use an induction argument on the order of derivatives. The base case $|\alpha| = 0$ has been treated in Section 3 and Section 4. We assume the proposition holds with order of derivatives on μ at most $|\alpha|$. For $|\alpha| + 1$, by commuting $Z_i^{\alpha+1}$ with $\underline{L}\mu = m + \mu e$, we have

$$\underline{L}\delta^l Z_i^{\alpha+1}\mu = (e + {}^{(Z_i)}\underline{Z})\delta^l Z_i^{\alpha+1}\mu + \delta^l Z_i^{\alpha+1}m + \sum_{\substack{|\beta_1|+|\beta_2|\leq|\alpha|+1 \\ |\beta_1|<|\alpha|}} \delta^{l_1} Z_i^{\beta_1}\mu \delta^{l_2} Z_i^{\beta_2}e$$

where $l_a, a = 1, 2$ is the number of T 's in Z^{β_a} 's and l is the number of T 's in Z^{α} 's. By the induction hypothesis, the above equation can be written as:

$$\underline{L}\delta^l Z_i^{\alpha+1}\mu = \mathcal{O}_2^{\leq 1} \cdot \delta^l Z_i^{\alpha+1}\mu + \Psi_0^{\leq |\alpha|+2}$$

Similar to the estimates derived in the Step 3 in previous section, we can use induction hypothesis and Gronwall's inequality to conclude that $\|Z_i^{\alpha+1}\mu\|_{L^\infty(\Sigma_t)} \lesssim_M \delta^{-l}$. \square

7.2. L^2 estimates. N_{top} will be the total number of derivatives commuted with $\square_{\bar{g}}\psi = 0$. The highest order objects will be of order $N_{\text{top}} + 1$. In this subsection, based on (B.1) and (B.2), we will derive L^2 estimates on the objects of order $\leq N_{\text{top}}$ in terms of the L^2 norms of $Z_i^{\alpha+2}\psi \in \Psi_{1-2l}^{|\alpha|+2}$ with $|\alpha| \leq N_{\text{top}} - 1$. We start with the following lemma:

Lemma 7.3. *For a ψ which vanishes on \underline{C}_0 , we have*

$$\int_{S_{t,\underline{u}}} \psi^2 \lesssim \delta \int_{\Sigma_t^{\underline{u}}} (L\psi)^2 + \mu(\underline{L}\psi)^2, \quad \int_{\Sigma_t^{\underline{u}}} \psi^2 \lesssim \delta^2 \int_{\Sigma_t^{\underline{u}}} (L\psi)^2 + \mu(\underline{L}\psi)^2. \quad (7.3)$$

Proof. Since $\psi(t, \underline{u}, \theta) = \int_0^{\underline{u}} T\psi(t, \underline{u}', \theta) d\underline{u}'$, we have

$$\begin{aligned} \int_{S_{t,\underline{u}}} \psi^2 d\mu_{\mathcal{G}} &= \int_{S_{t,\underline{u}}} \left(\int_0^{\underline{u}} T\psi(t, \underline{u}', \theta) d\underline{u}' \right)^2 d\mu_{\mathcal{G}(t,\underline{u})} \\ &\lesssim \delta \int_{S_{t,\underline{u}}} \int_0^{\underline{u}} (T\psi(t, \underline{u}', \theta))^2 d\underline{u}' d\mu_{\mathcal{G}(t,\underline{u})} \\ &\lesssim \delta \int_{S_{t,\underline{u}}} \int_0^{\underline{u}} (T\psi(t, \underline{u}', \theta))^2 d\mu_{\mathcal{G}(t,\underline{u}')} d\underline{u}' \end{aligned}$$

Here we have used the fact: $\sqrt{\det \mathcal{G}(t, \underline{u})} \lesssim \sqrt{\det \mathcal{G}(t, \underline{u}')} \lesssim \sqrt{\det \mathcal{G}(t, \underline{u})}$ due to the bound of the second fundamental form θ . On the other hand, $(T\psi) \lesssim (L\psi)^2 + \mu^2(\underline{L}\psi)^2$ and $\mu \lesssim 1$, the first inequality follows immediately. The second is an immediate consequence of the first one. \square

As a corollary, for $k \leq N_{\text{top}} - 1$, we have

$$\sum_{|\beta| \leq k} \int_{S_{t,\underline{u}}} (R_i^\beta \psi)^2 \lesssim \delta E_{\leq k+1}(t, \underline{u}). \quad (7.4)$$

Proposition 7.4. *For sufficiently small δ , for all α with $|\alpha| \leq N_{\text{top}} - 1$ and $t \in [-r_0, s^*]$, the $L^2(\Sigma_t^{\underline{u}})$ norms of all the quantities listed below*

$$\mathcal{L}_{Z_i}^\alpha \underline{X}', \mathcal{L}_{Z_i}^\alpha(Z_j) \underline{Z}, \mathcal{L}_{Z_i}^\alpha(Z_j) \not{X}, Z_i^{\alpha+1} y^j, Z_i^{\alpha+1} \lambda_j,$$

*are bounded*⁶ *by $\delta^{1/2-l} \int_{-r_0}^t \mu_m^{-1/2}(t') \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt'$, where l is the number of T 's in Z_i 's.*

Proof. We do induction on the order. When $|\alpha| = 0$, the result follows from the estimates in Section 3 and 4. Again, here we only treat the case $l = 0$. By assuming the proposition holds for terms with order $\leq |\alpha|$, we show it holds for $|\alpha| + 1$.

Step 1 Bounds on $\mathcal{L}_{R_i}^\alpha \underline{X}'$. By affording a \underline{L} -derivative, we have

$$\|\mathcal{L}_{R_i}^\alpha \underline{X}'\|_{L^2(\Sigma_t^{\underline{u}})} \lesssim \|\mathcal{L}_{R_i}^\alpha \underline{X}'\|_{L^2(\Sigma_{-r_0}^{\underline{u}})} + \int_{-r_0}^t \|\underline{X}'\| \|\mathcal{L}_{R_i}^\alpha \underline{X}'\| + \|\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{X}'\|_{L^2(\Sigma_{\tau}^{\underline{u}})} d\tau. \quad (7.5)$$

We use formula (7.2) to replace $\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{X}'$ by the terms with lower orders. Each nonlinear term has at most one factor with order $> N_\infty$. We bound this factor in $L^2(\Sigma_t)$ and the rest in L^∞ . We now indicate briefly how the estimates on the factors involving e and $\underline{\alpha}'$ work.

⁶The inequality is up to a constant depending only on the bootstrap constant M .

For $\underline{\alpha}'$, since

$$\underline{\alpha}'_{AB} = c \frac{dc}{d\rho} \mathcal{P}_{A,B}^2 \rho + \frac{1}{2} \left[\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2c^2} \left(\frac{dc^2}{d\rho} \right)^2 \right] X_A(\rho) X_B(\rho),$$

in view of the definition of $\underline{E}(t, \underline{u})$, for sufficiently small δ , we have

$$\|\mathcal{L}_{R_i}^\alpha \underline{\alpha}'_{AB}\|_{L^2(\Sigma_t^\underline{u})} \lesssim \sum_{|\alpha| \leq k} \|\mathcal{L} R_i^{\alpha+1} \psi\|_{L^2(\Sigma_t^\underline{u})} \lesssim_M \delta^{\frac{1}{2}} \mu_m^{-1/2}(t) \sqrt{\underline{E}_{\leq |\alpha|+2}(t, \underline{u})}.$$

For e , since $e = c^{-1} \frac{dc}{d\rho} \underline{L}\rho$, we have

$$\begin{aligned} \|R_i^\alpha e\|_{L^2(\Sigma_t^\underline{u})} &\lesssim_M \sum_{|\beta| \leq |\alpha|} \left(\delta^{1/2} \|\mathcal{L} R_i^{|\beta|-1} \psi\|_{L^2(\Sigma_t^\underline{u})} + \delta^{1/2} \|\mathcal{L} R_i^{|\beta|-1} \underline{L}\psi\|_{L^2(\Sigma_t^\underline{u})} \right) \\ &\lesssim_M \delta^{1/2} \mu_m^{-1/2}(t) \sqrt{\underline{E}_{\leq |\alpha|+1}(t, \underline{u})}. \end{aligned}$$

By applying Gronwall's inequality to (7.5), we obtain immediately that

$$\|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^2(\Sigma_t^\underline{u})} \lesssim_M \delta^{1/2} \int_{-r_0}^t \mu_m^{-1/2}(\tau) \sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau$$

Step 2 Bounds on $R_i^{\alpha+1} y^j$. By the computations in the Step 2 of the proof of Proposition 7.1, $R_i^{\alpha+1} y$ is a linear combination of the terms such as $\mathcal{L}_{R_i}^{\beta_1} \underline{\chi} \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_j \cdot \mathcal{L} R_i^{\beta_4} x^j$ or as $\mathcal{L}_{R_i}^{\beta_1} \mathcal{G} \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_j \cdot \mathcal{L} R_i^{\beta_4} x^j$, where $|\beta_1| + |\beta_2| + |\beta_3| + |\beta_4| = |\alpha|$. Similarly, we bound all factors with order $\leq N_\infty$ by the L^∞ estimates in Proposition 7.1. By the induction hypothesis, this yields the bound on $R_i^{\alpha+1} y^j$ immediately. The estimates for other quantities follow from the estimates of $\underline{\chi}'$ and y^j . In this process, the terms like $R_i^\beta x^j$ and the leading term in $\mathcal{L}_{R_i} R_j$, which can be bounded by a constant C disregarding the order of the derivatives, are bounded in L^∞ . The rest terms in $\mathcal{L}_{R_i} R_j$, which depend on $\underline{\chi}'$ and y^j as well as their derivatives, are bounded in L^2 based on the L^2 estimates for $\mathcal{L}_{R_i}^\beta \underline{\chi}'$. \square

We also have L^2 estimates for derivatives of μ .

Proposition 7.5. *For sufficiently small δ , for all α with $|\alpha| \leq N_{\text{top}} - 1$ and $t \in [-r_0, s^*]$, we have*

$$\delta^l \|Z_i^{\alpha+1} \mu\|_{L^2(\Sigma_t)} \lesssim_M \delta^l \|Z_i^{\alpha+1} \mu\|_{L^2(\Sigma_{-r_0}^\underline{u})} + \delta^{1/2} \int_{-r_0}^t \sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})} + \mu_m^{-1/2}(\tau) \sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau.$$

Proof. According to the proof of Proposition 7.2, we have

$$\delta^l \underline{L} |Z_i^{\alpha+1} \mu| \lesssim \delta^l |Z_i^{\alpha+1} m| + \delta^l (|e| + |(R_i) \underline{Z}|) |Z_i^{\alpha+1} \mu| + \sum_{|\beta_1 + \beta_2| \leq |\alpha|} \delta^{l_1} |Z_i^{\beta_1} \mu| \delta^{l_2} |R_i^{\beta_2} e|.$$

Then the result follows in the same way as Proposition 7.2. \square

8. ESTIMATES ON TOP ORDER TERMS

The highest possible order of an object in the paper will be $N_{\text{top}} + 1$. The current section is devoted to the L^2 estimates of $\mathcal{L} R_i^\alpha \text{tr} \underline{\chi}$ and $Z_i^{\alpha+2} \mu$ with $|\alpha| = N_{\text{top}} - 1$.

8.1. Estimates on $\text{tr}\underline{\chi}$. We first sketch the idea of the proof. Since we deal with top order terms, we can not use the transport equation (2.35) directly as in the previous section (which loses one derivative). Roughly speaking, we derive an elliptic system coupled with a transport equation for $\widehat{\underline{\chi}}$ and $\not{d}\text{tr}\underline{\chi}$:

$$\underline{L}(\not{d}R_i^\alpha \text{tr}\underline{\chi}) = \widehat{\underline{\chi}} \cdot \nabla \mathcal{L}_{R_i}^\alpha \widehat{\underline{\chi}} + \cdots, \quad \text{div} \mathcal{L}_{R_i}^\alpha \widehat{\underline{\chi}} = \not{d}R_i^\alpha \text{tr}\underline{\chi} + \cdots.$$

The new idea is using elliptic estimates and rewriting the right hand side of the transport equations to avoid the loss of derivatives.

Given $\square_{\tilde{g}}\psi_0 = 0$, since $\rho = \psi_0^2$, we can derive a wave equation for ρ :

$$\square_g \rho = \frac{d(\log(c))}{d\rho} g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho + 2g^{\mu\nu} \partial_\mu \psi_0 \partial_\nu \psi_0 \quad (8.1)$$

Therefore, in the null frame, we can rewrite $\underline{\Delta}\rho$ as $\underline{\Delta}\rho = \mu^{-1} \underline{L}(L\rho) + \text{l.o.t.}$ where l.o.t. represents all the terms with order at most 1. On the other hand, according to the definition of $\underline{\alpha}_{AB}$, we can rewrite (2.35) as $\underline{L}\text{tr}\underline{\chi} = -\frac{1}{2} \frac{d(c^2)}{d\rho} \underline{\Delta}\rho + \text{l.o.t.}$ where the lower order terms l.o.t. standard for terms with order at most 1. By substituting to the previous expression on $\underline{\Delta}\rho$, we obtain:

$$\underline{L}(\mu \text{tr}\underline{\chi} - \check{f}) = 2\underline{L}\mu \text{tr}\underline{\chi} - \frac{1}{2} \mu (\text{tr}\underline{\chi})^2 - \mu |\widehat{\underline{\chi}}|^2 + \check{g}, \quad (8.2)$$

where $\check{f} = -\frac{1}{2} \frac{d(c^2)}{d\rho} L\rho$ and \check{g} is given by

$$\check{g} = \left(2 \left(\frac{d(c)}{d\rho} \right)^2 + c \frac{d^2(c)}{d\rho^2} \right) (L\rho L\rho - \mu |\not{d}\rho|^2) + 2c \frac{d(c)}{d\rho} \left((L\psi_0 \underline{L}\psi_0 - \mu |\not{d}\psi_0|^2) + \left(\frac{1}{4} \frac{\mu |\not{d}\rho|^2}{c^2} - \underline{\zeta}^A \not{d}_A \rho \right) \right).$$

We observe two main features of (8.2): the order of the righthand side terms are one less than that of the lefthand side; It is regular in μ , i.e. there is no μ^{-1} factor. In order to commute R_i 's with (8.2) and to control the α^{th} derivatives of $\not{d}(\text{tr}\underline{\chi})$, for a given multi-index α , we introduce

$$F_\alpha = \mu \not{d}(R_i^\alpha \text{tr}\underline{\chi}) - \not{d}(R_i^\alpha \check{f}).$$

For $\alpha = 0$ and $F = F_0 = \mu \not{d}\text{tr}\underline{\chi} - \not{d}\check{f}$, by commuting \not{d} with (8.2), we obtain

$$\not{d}\underline{L}F + (\text{tr}\underline{\chi} - 2\mu^{-1} \underline{L}\mu)F = \left(-\frac{1}{2} \text{tr}\underline{\chi} + 2\mu^{-1} \underline{L}\mu \right) \not{d}\check{f} - \mu \not{d}(|\widehat{\underline{\chi}}|^2) + g_0$$

with $g_0 = \not{d}\check{g} - \frac{1}{2} \text{tr}\underline{\chi} \not{d}(\check{f} - 2\underline{L}\mu) - (\not{d}\mu)(\underline{L}\text{tr}\underline{\chi} + |\widehat{\underline{\chi}}|^2)$. Similarly, for $|\alpha| \neq 0$, we first commute R_i^α and then commute \not{d} with (8.2). This leads to

$$\not{d}\underline{L}F_\alpha + (\text{tr}\underline{\chi} - 2\mu^{-1} \underline{L}\mu) F_\alpha = \left(-\frac{1}{2} \text{tr}\underline{\chi} + 2\mu^{-1} \underline{L}\mu \right) \not{d}(R_i^\alpha \check{f}) - \mu \not{d}(R_i^\alpha (|\widehat{\underline{\chi}}|^2)) + g_\alpha, \quad (8.3)$$

with g_α in the following schematic expression (by setting all the numerical constants to be 1):

$$\begin{aligned} g_\alpha &= \not{d}R_i^\alpha g_0 + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \not{d}R_i^{\beta_1} \not{d}R_i^{\beta_2} \underline{L}F_{\beta_2} + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \not{d}R_i^{\beta_1} \left((\mu R_i \text{tr}\underline{\chi} + R_i \underline{L}\mu + {}^{(R_i)}\underline{Z}\mu) \not{d}(R_i^{\beta_2} \text{tr}\underline{\chi}) \right) \\ &+ \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \not{d}R_i^{\beta_1} \left(R_i \mu [\not{d}\underline{L} \not{d}(R_i^{\beta_2} \text{tr}\underline{\chi}) + \text{tr}\underline{\chi} \not{d}(R_i^{\beta_2} \text{tr}\underline{\chi}) + \not{d}(R_i^{\beta_2} (|\widehat{\underline{\chi}}|^2))] + R_i \text{tr}\underline{\chi} \not{d}(R_i^{\beta_2} \check{f}) \right). \end{aligned}$$

We remark that F_α is of order $N_{\text{top}} + 1$ so that $|\alpha| = N_{\text{top}} - 1$.

We rewrite (2.33) as $\text{div} \widehat{\underline{\chi}} = \frac{1}{2} \not{d}\text{tr}\underline{\chi} - (\mu^{-1} \underline{\zeta} \cdot \widehat{\underline{\chi}} - \frac{1}{2} \mu^{-1} \underline{\zeta} \text{tr}\underline{\chi})$. By commuting $\widehat{\underline{\chi}}_{R_i}^\alpha$, we obtain the following schematic expression:

$$\text{div}(\widehat{\underline{\chi}}_{R_i}^\alpha) = \frac{1}{2} \not{d}(R_i^\alpha \text{tr}\underline{\chi}) + H_\alpha, \quad (8.4)$$

with $H_\alpha = (\mathcal{L}_{R_i} + \frac{1}{2}\text{tr}^{(R_i)}\not\partial)^\alpha (\mu^{-1}\underline{\zeta} \cdot \widehat{\chi} - \frac{1}{2}\mu^{-1}\underline{\zeta} \text{tr}\underline{\chi}) + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} (\mathcal{L}_{R_i} + \frac{1}{2}\text{tr}^{(R_i)}\not\partial)^{\beta_1} (\text{tr}^{(R_i)}\not\partial \cdot \not\partial(R_i^{\beta_2}\text{tr}\underline{\chi}) + (\text{div}^{(R_i)}\widehat{\not\partial}) \cdot \widehat{\mathcal{L}}_{R_i}^{\beta_2}\widehat{\chi}) + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} (\mathcal{L}_{R_i} + \frac{1}{2}\text{tr}^{(R_i)}\not\partial)^{\beta_1} (\widehat{\mathcal{L}}_{R_i}^{\beta_2}\widehat{\chi} \cdot \nabla \widehat{\mathcal{L}}_{R_i}^{\beta_2}\widehat{\chi})$. By applying the elliptic estimates (3.44) to (8.4), in view of the definition of F_α , we obtain (the L^2 norms are taken at $S(t, \underline{u})$)

$$\|\mu \nabla \widehat{\mathcal{L}}_{R_i}^\alpha \widehat{\chi}\|_{L^2(S(t, \underline{u}))} \lesssim \|F_\alpha\|_{L^2} + \|\not\partial R_i^\alpha \check{f}\|_{L^2} + \|\not\partial \mu\|_{L^\infty} \|\widehat{\mathcal{L}}_{R_i}^\alpha \widehat{\chi}\|_{L^2} + \|\mu H_\alpha\|_{L^2}. \quad (8.5)$$

For any form ξ , since $|\xi|L|\xi| = (\xi, \mathcal{L}_L \xi) - \xi \cdot \widehat{\chi} \cdot \xi - \frac{1}{2}\text{tr}\underline{\chi}|\xi|^2$, we have $L|\xi| \leq |\mathcal{L}_L \xi| + |\widehat{\chi}||\xi| - \frac{1}{2}\text{tr}\underline{\chi}|\xi|$. Applying this inequality to (8.3), we obtain

$$L|F_\alpha| \leq (\mu^{-1}L\mu - \frac{3}{2}\text{tr}\underline{\chi} + |\widehat{\chi}|)|F_\alpha| + (2\mu^{-1}|L\mu| - \text{tr}\underline{\chi})|\not\partial R_i^\alpha \check{f}| + |\mu \not\partial R_i^\alpha (|\widehat{\chi}|^2)| + |g_\alpha|. \quad (8.6)$$

To obtain the $L^2(\Sigma_t)$ bound on F_α , we first integrate (8.6) from $-r_0$ to t and then take the L^2 norm on Σ_t . We claim that we can ignore the first term on the right hand side: The term $(|\text{tr}\underline{\chi}| + |\widehat{\chi}|)|F_\alpha|$ can be removed immediately by Gronwall's inequality. For $\mu^{-1}L\mu \cdot |F_\alpha|$, for a fixed (\underline{u}, θ) , if $\mu \geq \frac{1}{10}$ for all \underline{u} , it can be also removed by Gronwall's inequality; otherwise, $\mu < \frac{1}{10}$, therefore, according to the argument in Proposition 4.6, the sign of $L\mu$ is negative so that this term can be ignored. As a result, (8.6) yields

$$\begin{aligned} \|F_\alpha\|_{L^2(\Sigma_t)} &\lesssim \|F_\alpha\|_{L^2(\Sigma_{-r_0})} + \int_{-r_0}^t \|(\mu^{-1}|L\mu| + 1)\not\partial R_i^\alpha \check{f}\|_{L^2(\Sigma_\tau)} + \|\mu \not\partial R_i^\alpha (|\widehat{\chi}|^2)\|_{L^2(\Sigma_\tau)} + \|g_\alpha\|_{L^2(\Sigma_\tau)} d\tau \\ &= \|F_\alpha\|_{L^2(\Sigma_{-r_0})} + I_1 + I_2 + I_3. \end{aligned}$$

where the I_i 's are defined in the obvious way.

We first bound I_2 . According to the Leibniz rule, we have

$$I_2 = \sum_{|\beta_1|+|\beta_2|+|\beta_3|+|\beta_4|=|\alpha|} \int_{-r_0}^t \|\mu \not\partial (\widehat{\mathcal{L}}_{R_i}^{\beta_1} \not\partial \cdot \widehat{\mathcal{L}}_{R_i}^{\beta_2} \not\partial \cdot \widehat{\mathcal{L}}_{R_i}^{\beta_3} \widehat{\chi} \cdot \widehat{\mathcal{L}}_{R_i}^{\beta_4} \widehat{\chi})\|_{L^2(\Sigma_\tau)} d\tau.$$

Therefore, at least three indices of the β_i 's are at most N_∞ . According to Proposition 7.4, we have

$$\begin{aligned} I_2 &\lesssim \int_{-r_0}^t \|\mu \not\partial \widehat{\mathcal{L}}_{R_i}^\alpha \widehat{\chi}\|_{L^2(\Sigma_\tau)} \|\widehat{\chi}\|_{L^\infty(\Sigma_\tau)} + \delta^{3/2} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau \\ &\stackrel{(8.5)}{\lesssim} \delta \int_{-r_0}^t \|F_\alpha\|_{L^2(\Sigma_\tau)} + \|\not\partial R_i^\alpha \check{f}\|_{L^2(\Sigma_\tau)} + \delta^{1/2} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau. \end{aligned}$$

Since $\check{f} = -\frac{1}{2} \frac{dc^2}{d\rho} L\rho$, we have $\not\partial R_i^\alpha \check{f} = \frac{dc^2}{d\rho} \psi_0 \not\partial R_i^\alpha L\psi_0 + \not\partial \left(\sum_{\substack{\beta_1+\beta_2=\alpha, \\ |\beta_1| \geq 1}} R_i^{\beta_1} \left(\frac{dc^2}{d\rho} \psi_0 \right) R_i^{\beta_2} (L\psi_0) \right)$. This leads to

$$\int_{-r_0}^t \|\not\partial R_i^\alpha \check{f}\|_{L^2(\Sigma_\tau)} \lesssim \int_{-r_0}^t \delta^{\frac{1}{2}} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau.$$

Therefore, we have the following bound for I_2 :

$$I_2 \lesssim \int_{-r_0}^t \delta \|F_\alpha\|_{L^2(\Sigma_\tau)} + \delta^{3/2} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{3/2} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau. \quad (8.7)$$

We remark that, as long as the terms under consideration are not top order terms, i.e. not a term of order $N_{\text{top}} + 1$, we can simply use the estimates from previous section to get the estimates. The reason is as follows: each term is a product of $\mathcal{O}_k^{\leq l}$ with $l \leq |\alpha|$. Only one of the factor is of order $l > N_\infty$. We can

bound the rest in L^∞ and the highest order one in L^2 , thanks to the estimates derived in the previous section.

To bound I_3 , as we pointed out above, we only have to take care of the terms appearing in g_α whose orders are possibly $N_{\text{top}} + 1$. The rest of them can be easily bounded in L^2 by a universal constant times the sum of $\int_{-r_0}^t \delta^{\frac{1}{2}} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{1/2} \mu_m^{-1/2}(t) \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} d\tau$.

We now investigate the possible highest order terms in g_α . There are three possibilities: the first one are the terms of the form $\sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \mathfrak{L}_{R_i}^{\beta_1} \mathfrak{L}_{(R_i)} \underline{z} F_{\beta_2}$. They can be bounded by $|^{(R_i)} \underline{z}| |F_\alpha| \lesssim \delta^{1/2} |F_\alpha|$

provided that δ is suitably small. The second possibility is from the $(\underline{L} \text{tr} \underline{\chi} + |\underline{\chi}|^2)$ term of g_0 . However, equation (2.35) says that $\underline{L} \text{tr} \underline{\chi} + |\underline{\chi}|^2 = e - \text{tr} \alpha'$, so although it is of the highest order, the highest order part consists only ∇ derivatives of ψ (thanks to the expression of α'), hence can be bounded in the same way as lower order terms. The last possibility is from the term $\underline{L} \psi_0 L \psi_0$ appearing in \check{g} . They are of top orders and they can not be converted into terms involving only ∇ derivatives. These terms contribute to g_α the terms of the form $\mathcal{O}_0^{\leq 1} \cdot L \psi_0 \cdot \underline{L} R_i^{\alpha+1} \psi_0$ in g_α .

$$\begin{aligned} \int_{-r_0}^t \|L \psi_0 \underline{L} R_i^{\alpha+1} \psi_0\|_{L^2(\Sigma_\tau^{\underline{u}})} d\tau &\lesssim \delta^{-1/2} \left(\int_{-r_0}^t \int_0^{\underline{u}} \int_{S_{\tau, \underline{u}'}} (\underline{L} R_i^{\alpha+1} \psi)^2 d\mu_{\check{g}} d\underline{u}' d\tau \right)^{1/2} \\ &\lesssim \delta^{-1/2} \left(\int_0^{\underline{u}} \underline{E}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2} \end{aligned}$$

Therefore, we have the following estimates for I_3 :

$$I_3 \lesssim \int_{-r_0}^t \delta^{\frac{1}{2}} \sqrt{E_{\leq |\alpha|+2}(\tau)} + \delta^{1/2} \mu_m^{-1/2}(\tau) \sqrt{E_{\leq |\alpha|+2}(t)} d\tau + \delta^{-1/2} \left(\int_0^{\underline{u}} \underline{E}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2} \quad (8.8)$$

We now study the estimates on I_1 . When $\mu \geq \frac{1}{10}$, the estimates are straightforward. To study the case when $\mu \leq \frac{1}{10}$, we introduce a few notations (where a is a positive constant):

$$t_0 = \inf \left\{ \tau \in [-2, t^*) \mid \mu_m(t) < \frac{1}{10} \right\}, \quad M(t) = \max_{(\underline{u}, \theta) \in W_{\text{shock}}} |(\underline{L}(\log \mu))_-(t, \underline{u}, \theta)|, \quad I_a(t) = \int_{t_0}^t \mu_m^{-a}(\tau) M(\tau) d\tau.$$

Lemma 8.1. *We assume that a is at least 4.*

(1) *For sufficiently large a and for all $t \in [t_0, t^*)$, we have*

$$I_a(t) \lesssim a^{-1} \mu_m^{-a}(t). \quad (8.9)$$

(1') *For sufficiently large a and for all $t \in [t_0, t^*)$, we have*

$$\int_{t_0}^t \mu_m^{-a-1}(t') dt' \lesssim \frac{1}{a} \mu_m^{-a}(t)$$

(2) *For $a \geq 4$ and sufficiently small δ , there is a constant C_0 independent of a and δ , so that for all $\tau \in [-r_0, t]$, we have*

$$\mu_m^a(t) \leq C_0 \mu_m^a(\tau) \quad (8.10)$$

Proof. (1) By Proposition 4.6, for $t \geq t_0$, the minimum of $r_0^2(\underline{L}\mu)(-r_0, \underline{u}, \theta)$ on $[0, \delta] \times \mathbb{S}^2$ is negative and we denote it by

$$-\eta_m = \min_{(\underline{u}, \theta) \in [0, \delta] \times \mathbb{S}^2} \{r_0^2(\underline{L}\mu)(-r_0, \underline{u}, \theta)\}. \quad (8.11)$$

We notice that $1 \leq \eta_m \leq C_m$ where C_m is a constant depending on the initial data. In view of the asymptotic expansion for $(\underline{L}\mu)(t, \underline{u}, \theta)$ in Lemma 4.4, we have

$$\mu(t, \underline{u}, \theta) = 1 - \left(\frac{1}{t} + \frac{1}{r_0} \right) r_0^2 (\underline{L}\mu)(-r_0, \underline{u}, \theta) + O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2} \right). \quad (8.12)$$

We fix an $s \in (t_0, t^*)$ in such a way that $t_0 \leq t < s < t^*$. There exists $(\underline{u}_s, \theta_s) \in [0, \delta] \times \mathbb{S}^2$ and $(\underline{u}_m, \theta_m) \in [0, \delta] \times \mathbb{S}^2$ so that

$$\mu(s, \underline{u}_s, \theta_s) = \mu_m(s), \quad r_0^2 (\underline{L}\mu)(-r_0, \underline{u}_m, \theta_m) = -\eta_m. \quad (8.13)$$

We claim that

$$|\eta_m + r_0^2 (\underline{L}\mu)(-r_0, \underline{u}_s, \theta_s)| \leq O(\delta M^4). \quad (8.14)$$

Indeed, one can apply (8.12) to $\mu(s, \underline{u}_m, \theta_m)$ and $\mu(s, \underline{u}_s, \theta_s)$ to derive

$$\begin{aligned} \mu(s, \underline{u}_s, \theta_s) &= 1 - \left(\frac{1}{s} + \frac{1}{r_0} \right) (-\eta_m + d_{ms}) + O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2} \right) \\ \mu(s, \underline{u}_m, \theta_m) &= 1 - \left(\frac{1}{s} + \frac{1}{r_0} \right) (-\eta_m) + O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2} \right), \end{aligned} \quad (8.15)$$

where the quantity $d_{ms} > 0$ is defined as

$$d_{ms} := \eta_m + r_0^2 (\underline{L}\mu)(-r_0, \underline{u}_s, \theta_s). \quad (8.16)$$

Since $\mu(s, \underline{u}_s, \theta_s) \leq \mu(s, \underline{u}_m, \theta_m)$, we have

$$0 < - \left(\frac{1}{s} + \frac{1}{r_0} \right) d_{ms} \leq O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2} \right).$$

Hence,

$$d_{ms} \leq O(\delta M^4). \quad (8.17)$$

The constants in the above inequalities depend on t_0 therefore on η_m and they are absolute constants. With this preparation, one can derive precise upper and lower bounds for $\mu_m(t)$.

We pick up a $(\underline{u}'_m, \theta'_m) \in [0, \delta] \times \mathbb{S}^2$ in such a way that $\mu(t, \underline{u}'_m, \theta'_m) = \mu_m(t)$. For the lower bound, by virtue of Lemma 4.4, we have

$$\begin{aligned} \mu_m(t) &= \mu(t, \underline{u}'_m, \theta'_m) = \mu(s, \underline{u}'_m, \theta'_m) + \int_s^t (\underline{L}\mu)(t', \underline{u}'_m, \theta'_m) dt' \\ &\geq \mu_m(s) + \int_s^t \frac{\eta_m}{-t'^2} + \frac{O(\delta M^4)}{(-t')^3} dt' \\ &\geq \mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) \left(\frac{1}{t} - \frac{1}{s} \right). \end{aligned} \quad (8.18)$$

In the last step, we take sufficiently small δ so that $O(\delta M^4) \leq \frac{1}{2a}$.

For the upper bound, in view of Lemma 4.4 and (8.17), we have

$$\begin{aligned}
\mu_m(t) &\leq \mu(t, \underline{u}_s, \theta_s) = \mu_m(s) + \int_s^t (\underline{L}\mu)(t', \underline{u}_s, \theta_s) dt' \\
&= \mu_m(s) + \int_s^t \frac{\eta_m - d_{ms}}{-t'^2} + \frac{O(\delta M^4)}{(-t')^3} dt' \\
&\leq \mu_m(s) + \int_s^t \frac{\eta_m}{-t'^2} + \frac{O(\delta M^4)}{t'^2} dt' \\
&\leq \mu_m(s) + \left(\eta_m + \frac{1}{2a} \right) \left(\frac{1}{t} - \frac{1}{s} \right)
\end{aligned} \tag{8.19}$$

In the last step, we also take sufficiently small δ so that $O(\delta M^4) \leq \frac{1}{2a}$.

For $I_a(t)$, first of all, we have

$$\begin{aligned}
I_a(t) &\lesssim \int_{t_0}^t \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) \left(\frac{1}{t} - \frac{1}{s} \right) \right)^{-a-1} t'^{-2} dt' \\
&= \int_{\tau}^{\tau_0} \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) (\tau - \tau_s) \right)^{-a-1} d\tau' \\
&\leq \frac{1}{\eta_m - \frac{1}{2a}} \frac{1}{a} \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) (\tau - \tau_s) \right)^{-a}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_a(t) &\lesssim \frac{1}{a} \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) \left(\frac{1}{t} - \frac{1}{s} \right) \right)^{-a} \\
&\leq \frac{1}{a} \frac{(\mu_m(s) + (\eta_m - \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}}{(\mu_m(s) + (\eta_m + \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}} \mu_m^{-a}(t) \\
&\leq \frac{1}{a} \frac{((\eta_m - \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}}{((\eta_m + \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}} \mu_m^{-a}(t).
\end{aligned} \tag{8.20}$$

Since as $a \rightarrow \infty$, one has

$$\frac{(\eta_m - \frac{1}{2a})^{-a}}{(\eta_m + \frac{1}{2a})^{-a}} \rightarrow e^{\frac{1}{\eta_m}}.$$

The limit is an absolute constant. Therefore, (8.20) yields the proof for part (1) of the lemma. The proof for part (1') is exactly the same.

(2) We start with an easy observation: if $\mu(t, \underline{u}, \theta) \leq 1 - \frac{1}{a}$, then $\underline{L}\mu(t, \underline{u}, \theta) \lesssim -a^{-1}$. In fact, we claim that $(\frac{1}{t} + \frac{1}{r_0})r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \geq \frac{1}{2}a^{-1}$. Otherwise, for sufficiently small δ (say $\delta^{1/4} \leq a^{-1}$), according to the expansion for $\mu(t, \underline{u}, \theta)$, i.e. $\mu(t, \underline{u}, \theta) = \mu(-r_0, \underline{u}, \theta) - (\frac{1}{t} + \frac{1}{r_0})r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) + O(\delta)$, we have

$$\mu(t, \underline{u}, \theta) > 1 - \frac{1}{2a} - C\delta \geq 1 - \frac{1}{a}.$$

which is a contradiction. Therefore Lemma 4.4 implies $\underline{L}\mu(t, \underline{u}, \theta) \lesssim -a^{-1}$. In particular, this observation implies that, if there is a $t' \in [-r_0, s^*]$, so that $\mu_m(t') \leq 1 - a^{-1}$, then for all $t \geq t'$, we have $\mu_m(t) \leq 1 - a^{-1}$. This allows us to define a time t_1 , such that it is the minimum of all such t' with $\mu_m(t') \leq 1 - a^{-1}$.

We now prove the lemma. If $\tau \leq t_1$, since $\mu_m(t) \leq 2$, we have

$$\mu_m^{-a}(\tau) \leq (1 - \frac{1}{a})^{-a} \leq C_0 \leq C_0 \mu_m^{-a}(t).$$

If $\tau \geq t_1$, then $\mu_m(\tau) \leq 1 - \frac{1}{a}$. Let $\mu_m(\tau) = \mu(\tau, \underline{u}_\tau, \theta_\tau)$. We know that $\mu(t, \underline{u}_\tau, \theta_\tau)$ is decreasing in t for $t \geq \tau$. Therefore, we have

$$\mu_m(t) \leq \mu(t, \underline{u}_\tau, \theta_\tau) \leq \mu(\tau, \underline{u}_\tau, \theta_\tau) = \mu_m(\tau).$$

The proof now is complete. \square

For I_1 , according to the above lemma with $a = b_{|\alpha|+2}$, we then have

$$\begin{aligned} I_1 &\lesssim \delta^{1/2} \sum_{\beta \leq |\alpha|} \int_{-r_0}^t \| |\underline{L}(\log \mu)| R_i^{\beta+1} L\psi \|_{L^2(\Sigma_\tau)} \\ &\lesssim \delta^{1/2} I_{b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ &\lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}. \end{aligned} \quad (8.21)$$

Finally, the estimates (8.21), (8.7), (8.8) on I_1 , I_2 and I_3 together yield

$$\begin{aligned} \|F_\alpha\|_{L^2(\Sigma_t)} &\lesssim \|F_\alpha\|_{L^2(\Sigma_{-r_0})} + \int_{-r_0}^t \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(\tau) \left(\sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} + \mu_m^{-1/2}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} \right) d\tau \\ &\quad + \delta^{-\frac{1}{2}} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\int_0^t \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}') du'} + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}. \end{aligned} \quad (8.22)$$

This implies

$$\begin{aligned} \|\mu \not{d}(R_i^\alpha \text{tr} \chi)\|_{L^2(\Sigma_t)} &\lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ &\quad + \int_{-r_0}^t \delta^{1/2} \mu_m^{-b_{|\alpha|+2}-1/2}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau + \delta^{-1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\int_0^t \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}') du'}. \end{aligned} \quad (8.23)$$

8.2. Estimates on μ . The top order estimates on μ depend on the equation $\underline{L}\mu = m + \mu e$. Since $m = -\frac{1}{2} \frac{d(c^2)}{d\rho} T\rho$ and $e\mu = \frac{1}{2c^2} \frac{d(c^2)}{d\rho} \underline{L}\rho \cdot \mu$, it is visible that μ can be bounded by the total energy on ψ , i.e. the E_k 's. However, to avoid loss of derivatives, we should not integrate $\underline{L}\mu$ directly.

In view of the following commutation formulas,

$$\begin{aligned} [\underline{L}, \not{D}] \phi + \text{tr} \chi \not{D} \phi &= -2 \widehat{\chi} \cdot \widehat{\not{D}}^2 \phi - 2 \text{div} \widehat{\chi} \cdot \not{D} \phi, \\ [T, \not{D}] \phi + c^{-1} \mu \text{tr} \theta \not{D} \phi &= -2c^{-1} \mu \widehat{\theta} \cdot \widehat{\not{D}}^2 \phi - 2 \text{div}(c^{-1} \mu \widehat{\theta}) \cdot \not{D} \phi, \end{aligned}$$

we have

$$\begin{aligned} \underline{L} \not{D} \mu &= -\frac{1}{2} \frac{dc^2}{d\rho} \not{D} T\rho + \mu \not{D} e + e \not{D} \mu \\ &\quad + \not{D} \mu \cdot \not{D} e - \text{tr} \chi \not{D} \mu - 2 \widehat{\chi} \cdot \widehat{\not{D}}^2 \mu - 2 \text{div} \widehat{\chi} \cdot \not{D} \mu. \end{aligned} \quad (8.24)$$

According to (8.1),

$$\square_{g\rho} = \frac{d \log(c)}{d\rho} (\mu^{-1} \underline{L}\rho L\rho + \not{D}\rho \cdot \not{D}\rho) + 2\mu^{-1} L\psi_0 \underline{L}\psi_0 + 2 \not{D}\psi_0 \cdot \not{D}\psi_0.$$

Therefore, by multiplying μ , we have

$$\mu \mathbb{A}\rho = \underline{L}(\underline{L}\rho) + \frac{1}{2}\underline{L}\rho\text{tr}\chi + \frac{1}{2}\underline{L}\rho\text{tr}\underline{\chi} + \frac{d\log(c)}{d\rho}(\underline{L}\rho\underline{L}\rho + \mu\mathbb{A}\rho \cdot \mathbb{A}\rho) + 2\underline{L}\psi_0\underline{L}\psi_0 + 2\mu\mathbb{A}\psi_0 \cdot \mathbb{A}\psi_0.$$

We commute T and obtain

$$\begin{aligned} \mu \mathbb{A}T\rho &= \underline{L}(T\underline{L}\rho) + \frac{1}{2}\underline{L}\rho(T\text{tr}\chi) + \frac{1}{2}\underline{L}\rho(T\text{tr}\underline{\chi}) \\ &\quad + \{T, \underline{L}\}\underline{L}\rho + \frac{1}{2}\text{tr}\chi T\underline{L}\rho + \frac{1}{2}T\underline{L}\rho\text{tr}\underline{\chi} \\ &\quad + T\left(\frac{d\log(c)}{d\rho}(\underline{L}\rho\underline{L}\rho + \mu\mathbb{A}\rho \cdot \mathbb{A}\rho) + 2\underline{L}\psi_0\underline{L}\psi_0 + 2\mu\mathbb{A}\psi_0 \cdot \mathbb{A}\psi_0\right) \\ &\quad + c^{-1}\mu\text{tr}\theta\mathbb{A}\rho + 2c^{-1}\mu\hat{\theta} \cdot \hat{\mathbb{D}}^2\rho + 2\text{div}(c^{-1}\mu\hat{\theta}) \cdot \mathbb{A}\rho - (T\mu)\mathbb{A}\rho. \end{aligned} \quad (8.25)$$

Therefore

$$\begin{aligned} -\frac{1}{2}\frac{dc^2}{d\rho}\mathbb{A}T\rho &= \underline{L}\left(-\frac{1}{2}\frac{dc^2}{d\rho}T\underline{L}\rho\right) - \frac{1}{2}\frac{dc^2}{d\rho}\left(\frac{1}{2}\underline{L}\rho(T\text{tr}\chi) + \frac{1}{2}\underline{L}\rho(T\text{tr}\underline{\chi})\right) \\ &\quad - \frac{1}{2}\frac{dc^2}{d\rho}\left(\{T, \underline{L}\}\underline{L}\rho + \frac{1}{2}\text{tr}\chi T\underline{L}\rho + \frac{1}{2}T\underline{L}\rho\text{tr}\underline{\chi}\right) \\ &\quad - \frac{1}{2}\frac{dc^2}{d\rho}T\left(\frac{d\log(c)}{d\rho}(\underline{L}\rho\underline{L}\rho + \mu\mathbb{A}\rho \cdot \mathbb{A}\rho) + 2\underline{L}\psi_0\underline{L}\psi_0 + 2\mu\mathbb{A}\psi_0 \cdot \mathbb{A}\psi_0\right) \\ &\quad - \frac{1}{2}\frac{dc^2}{d\rho}\left(c^{-1}\mu\text{tr}\theta\mathbb{A}\rho + 2c^{-1}\mu\hat{\theta} \cdot \hat{\mathbb{D}}^2\rho + 2\text{div}(c^{-1}\mu\hat{\theta}) \cdot \mathbb{A}\rho - (T\mu)\mathbb{A}\rho\right) \\ &\quad + \underline{L}\left(\frac{1}{2}\frac{dc^2}{d\rho}\right)T\underline{L}\rho. \end{aligned} \quad (8.26)$$

In view of the commutator formula, we also have

$$\begin{aligned} \mu^2 \mathbb{A}e &= \underline{L}\left(\frac{\mu^2}{2c^2}\frac{dc^2}{d\rho}\mathbb{A}\rho\right) + \frac{\mu^2}{c^2}\frac{dc^2}{d\rho}\left(\underline{\chi} \cdot \hat{\mathbb{D}}^2\rho + \text{div}\hat{\chi} \cdot \mathbb{A}\rho\right) - \underline{L}\left(\frac{\mu^2}{2c^2}\frac{dc^2}{d\rho}\right)\mathbb{A}\rho \\ &\quad + \mu^2\frac{d}{d\rho}\left(\frac{1}{c^2}\frac{dc^2}{d\rho}\right)\mathbb{A}\rho \cdot \mathbb{A}\underline{L}\rho + \mu^2\left(\frac{d}{d\rho}\left(\frac{1}{2c^2}\frac{dc^2}{d\rho}\right)\mathbb{A}\rho + \frac{d^2}{d\rho^2}\left(\frac{1}{2c^2}\frac{dc^2}{d\rho}\right)|\mathbb{A}\rho|^2\right)\underline{L}\rho. \end{aligned} \quad (8.27)$$

Let us define

$$\check{f}' := -\frac{1}{2}\frac{dc^2}{d\rho}T\underline{L}\rho + \frac{\mu^2}{2c^2}\frac{dc^2}{d\rho}\mathbb{A}\rho, \quad F' := \mu\mathbb{A}\mu - \check{f}'. \quad (8.28)$$

Then in view of (8.24), (8.26) and (8.27), we obtain the following propagation equation for F' :

$$\underline{L}F' + (\text{tr}\underline{\chi} - 2\mu^{-1}\underline{L}\mu)F' = -\left(\frac{1}{2}\text{tr}\underline{\chi} - 2\mu^{-1}\underline{L}\mu\right)\check{f}' - 2\mu\underline{\chi} \cdot \hat{\mathbb{D}}^2\mu + \check{g}'. \quad (8.29)$$

where

$$\check{g}' = \left(-\mathbb{A}\mu + \frac{\mu}{c^2}\frac{dc^2}{d\rho}\mathbb{A}\rho\right) \cdot (\mu\mathbb{A}\text{tr}\underline{\chi}) + \Psi_{\geq -2}^{\leq 2} + \mathcal{O}_0^{\leq 1}\Psi_{\geq -2}^{\leq 2} + \Psi_{\geq 0}^{\leq 2}. \quad (8.30)$$

Here we have used the structure equation (2.34) to cancel the contribution from the term $\frac{1}{2}\underline{L}\rho(T\text{tr}\chi) + \frac{1}{2}\underline{L}\rho(T\text{tr}\underline{\chi})$ in (8.25) and the term $(\underline{L}\mu)\mathbb{A}\mu$ when we write $\underline{L}(\mu\mathbb{A}\mu) = \mu\underline{L}(\mathbb{A}\mu) + \underline{L}(\mu)\mathbb{A}\mu$. We also remark that the L^2 norm of all derivatives on $\text{div}\hat{\chi}$ has been estimated from previous subsection. In such a sense, it can also be considered as a $\Psi_2^{\leq 2}$ term and we use (2.33) to replace $\text{div}\hat{\chi}$ by $\mathbb{A}\text{tr}\underline{\chi} + \dots$. The term $\Psi_{\geq -2}^{\leq 2}$

comes from the contribution of $L\psi_0\bar{L}\psi_0$ and $\mathcal{O}_0^{\leq 1}\Psi_{\geq -2}^{\leq 2}$ comes from $-\frac{1}{4}\frac{dc^2}{d\rho}TL\rho\text{tr}\chi$ in (8.26). Since we already applied T to $L\psi_0\bar{L}\psi_0$ once in (8.26), instead of using flux $\bar{F}(t, \underline{u})$ as we did in the last subsection, we only need to use the energy $E(t, \underline{u})$ to control the contribution of this term.

We set $F'_{\alpha, l} = \mu R_i^{\alpha'} T^l \bar{\Delta}\mu - R_i^{\alpha'} T^l \check{f}'$ and $|\alpha'| + l = |\alpha|$. According to (8.29), we have

$$\bar{L}F'_{\alpha, l} + (\text{tr}\chi - 2\mu^{-1}\bar{L}\mu) F'_{\alpha, l} = -\left(\frac{1}{2}\text{tr}\chi - 2\mu^{-1}\bar{L}\mu\right) R_i^{\alpha'} T^l \check{f}' - 2\mu\widehat{\chi} \cdot \check{\mathcal{L}}_{R_i}^{\alpha'} \check{\mathcal{L}}_T^l \widehat{\bar{D}}^2 \mu + \check{g}'_{\alpha', l}. \quad (8.31)$$

where $\check{g}'_{\alpha', l}$ is given by

$$\begin{aligned} \check{g}'_{\alpha', l} = & \left(-\check{\mathcal{L}}\mu + \frac{\mu}{c^2}\frac{dc^2}{d\rho}\check{\mathcal{L}}\rho\right) \cdot \mu\check{\mathcal{L}}\left(R_i^{\alpha'} T^l \text{tr}\chi\right) + l \cdot \Lambda F'_{\alpha, l-1} + {}^{(R_i)}Z F'_{\alpha-1, l} \\ & + \mathcal{O}_{-k}^{\leq |\alpha'|+1} \mathcal{O}_{\geq -2l+k}^{\leq |\alpha|-|\alpha'|+1} + \Psi_{\geq -2l-2}^{\leq |\alpha|+2} + \mathcal{O}_{-k}^{\leq |\alpha'|+1} \Psi_{\geq -2l-2+k}^{\leq |\alpha|+2-|\alpha'|} + \Psi_{\geq -2l}^{\leq |\alpha|+2}. \end{aligned}$$

We remark that the second term of $\check{g}'_{\alpha', l}$ vanishes when $l = 0$ and $\Lambda = [\bar{L}, T]$. According to (8.31), we have

$$\begin{aligned} \|F'_{\alpha, l}\|_{L^2(\Sigma_t)} & \lesssim \|F'_{\alpha, l}\|_{L^2(\Sigma_{-r_0})} + \int_{-r_0}^t \|\mu^{-1}\bar{L}\mu\|_{L^\infty(\Sigma_\tau)} \|R_i^{\alpha'} T^l \check{f}'\|_{L^2(\Sigma_\tau)} d\tau \\ & \quad + \int_{-r_0}^t \|\mu\widehat{\chi} \cdot \check{\mathcal{L}}_{R_i}^{\alpha'} \check{\mathcal{L}}_T^l \widehat{\bar{D}}^2 \mu\|_{L^2(\Sigma_\tau)} d\tau + \int_{-r_0}^t \|\check{g}'_{\alpha', l}\|_{L^2(\Sigma_\tau)} d\tau \\ & = \|F'_{\alpha}\|_{L^2(\Sigma_{-r_0})} + I_1 + I_2 + I_3. \end{aligned}$$

We remark that we must multiply both sides δ^l to get the correct estimates. We first deal with I_1 . Since $\check{f}' = \frac{1}{2}\frac{dc^2}{d\rho}\left(-TL\rho + \frac{\mu^2}{c^2}\bar{\Delta}\rho\right)$, therefore, we have

$$R_i^{\alpha'} T^l \check{f}' = -\frac{dc^2}{d\rho}\psi_0 R_i^{\alpha'} T^{l+1} L\psi_0 + \frac{\mu^2}{2c^2}\frac{dc^2}{d\rho}\psi_0 R_i^{\alpha'} T^l \bar{\Delta}\psi_0 + \Psi_{\geq -2(l+2)}^{\leq |\alpha|+2}.$$

Compared to the first two terms, the last term on the right hand side above is of lower order with respect to the order of derivatives. Hence,

$$\delta^{l+1}\|R_i^{\alpha'} T^l \check{f}'\|_{L^2(\Sigma_\tau)} \lesssim \delta^{1/2}\sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{3/2}\sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})} \quad (8.32)$$

This together with Lemma 8.1 yields

$$\delta^{l+1}I_1 \lesssim \mu_m^{-b_{|\alpha|+2}}(t)\delta^{1/2}\left(\sqrt{\widetilde{E}_{\leq |\alpha|+2}(t, \underline{u})} + \delta\sqrt{\widetilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})}\right). \quad (8.33)$$

For I_2 , we can use elliptic estimates, i.e. to bound $\widehat{\bar{D}}^2 \mu$ by $\bar{\Delta}\mu$. This leads to

$$\begin{aligned} \delta^{l+1}I_2 & \lesssim \delta^{l+2} \int_{-r_0}^t \|\mu R_i^{\alpha'} T^l \bar{\Delta}\mu\|_{L^2(\Sigma_\tau)} d\tau \\ & \quad + \int_{-r_0}^t \delta^{3/2}\sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{3/2}\mu_m^{-1/2}(\tau)\sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau \\ & \lesssim \delta \int_{-r_0}^t \delta^{l+1}\|F_{\alpha, l}\|_{L^2(\Sigma_\tau)} + \delta^{l+1}\|R_i^{\alpha'} T^l \check{f}'\|_{L^2(\Sigma_\tau)} \\ & \quad + \delta^{1/2}\left(\sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \mu_m^{-1/2}(\tau)\sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})}\right) d\tau. \end{aligned} \quad (8.34)$$

We can skip the first two terms: The second term is already controlled by I_1 . While the first term will be eventually absorbed by Gronwall's inequality. The last two terms come from the commutator between $\underline{\Delta}$ and T, R_i as well as using Proposition 7.5.

For $I_3 := I_3' + I_3''$, we first consider the contributions from the first line in the expression of $\check{g}'_{\alpha', l}$, which are denoted by I_3' . We consider the cases $l = 0$ and $l > 0$ separately. For $l = 0$, we have

$$\begin{aligned} \delta I_3' &\lesssim \delta \int_{-r_0}^t \left\| \left(-\not{d}\mu + \frac{\mu}{c^2} \frac{dc^2}{d\rho} \not{d}\rho \right) (\mu \not{d}(R_i^\alpha \text{tr}\underline{\chi})) + \left(|^{(R_i)}\underline{Z}| + |\Lambda| \right) \not{d}F'_{|\alpha|-1} \right\|_{L^2(\Sigma_\tau)} d\tau \\ &\lesssim \delta \int_{-r_0}^t \left\| \mu \not{d}(R_i^\alpha \text{tr}\underline{\chi}) \right\|_{L^2(\Sigma_\tau)} + \delta \|\not{d}F'_{|\alpha|-1}\|_{L^2(\Sigma_\tau)} d\tau \\ &\quad + \int_{-r_0}^t \left(\delta^{3/2} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{3/2} \mu_m^{-1/2}(\tau) \sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})} \right) d\tau \end{aligned}$$

The first two terms are bound by the top order estimates on $\text{tr}\underline{\chi}$ in the previous subsection, therefore, for $l = 0$, we have

$$\begin{aligned} \delta I_3' &\lesssim \int_{-r_0}^t \delta^{3/2} \mu_m^{-b_{|\alpha|+2}}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{3/2} \mu_m^{-b_{|\alpha|+2}-1/2}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau \\ &\quad + \delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(\tau) \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u}') d\underline{u}'} d\tau. \end{aligned} \quad (8.35)$$

For the $l \geq 1$ case, we use (2.34), i.e. $\not{L}T\underline{\chi} = \nabla \widehat{\otimes} \underline{\eta} + \mu^{-1} \underline{\zeta} \widehat{\otimes} \underline{\eta} - c^{-1} \underline{L}(c^{-1} \mu) \underline{\chi} + c^{-1} \mu \theta \widehat{\otimes} \underline{\chi}$ to rewrite $T \text{tr}\underline{\chi}$. By taking the trace in (2.34), we obtain

$$\mu \not{d}(R_i^{\alpha'} T^l \text{tr}\underline{\chi}) = \mu R_i^{\alpha'} T^{l-1} \not{d}\mu + \mathcal{O}_{\geq -2l+2}^{\leq |\alpha|+1},$$

where

$$\delta^{l+1} \|\mathcal{O}_{\geq -2l+2}^{\leq |\alpha|+1}\|_{L^2(\Sigma_\tau)} \lesssim \delta^{3/2} \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} + \delta^{3/2} \mu_m^{-1/2}(\tau) \sqrt{\underline{E}_{\leq |\alpha|+2}(\tau, \underline{u})}.$$

We then conclude that (for $l \geq 1$)

$$\delta^{l+1} I_3' \lesssim \int_{-r_0}^t \delta^{3/2} \mu_m^{-b_{|\alpha|+2}}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} + \delta^{3/2} \mu_m^{-b_{|\alpha|+2}-\frac{1}{2}}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau. \quad (8.36)$$

Now we discuss the contributions from the last three terms in the expression of $\check{g}'_{\alpha', l}$, which are denoted by I_3'' . In view of Proposition 7.4 and 7.5, the terms $\mathcal{O}_{-k}^{\leq |\alpha'|+1} \mathcal{O}_{2-k}^{\leq |\alpha|-|\alpha'|+1}$ can be absorbed by $\mathcal{O}_{-k}^{\leq |\alpha'|+1} \Psi_{\geq -2l-2+k}^{\leq |\alpha|+2-|\alpha'|}$. We can bound the last three terms of $\check{g}'_{\alpha', l}$ as follows

$$\begin{aligned} \delta^{l+1} \|\Psi_{\geq -2l-2}^{\leq |\alpha|+2}\|_{L^2(\Sigma_t)} &\lesssim \delta^{1/2} \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})}, \\ \delta^{l+1} \|\mathcal{O}_{-k}^{\leq |\alpha'|+1} \Psi_{\geq -2l-2+k}^{\leq |\alpha|+2-|\alpha'|}\|_{L^2(\Sigma_t^{\underline{u}})} &\lesssim \delta^{1/2} \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} + \delta^{1/2} \int_{-r_0}^t \mu_m^{-1/2}(t') \sqrt{\underline{E}_{\leq |\alpha|+2}(t', \underline{u})} dt', \\ \delta^{l+1} \|\Psi_{\geq -2l}^{\leq |\alpha|+2}\|_{L^2(\Sigma_t)} &\lesssim \delta^{3/2} \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} + \delta^{3/2} \mu_m^{-1/2}(t) \sqrt{\underline{E}_{\leq |\alpha|+2}(t, \underline{u})}. \end{aligned} \quad (8.37)$$

Therefore we have the following estimates for I_3'' :

$$\delta^{l+1} I_3'' \lesssim \delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' + \delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt'. \quad (8.38)$$

By combining the estimate (8.32) for \check{f}' and the estimates for (8.33), (8.34), (8.35), (8.36) and (8.38) for I_1, I_2, I_3 , we obtain

$$\begin{aligned}
\delta^{l+1} \|R_i^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_t)} &\lesssim \delta^{l+1} \|F_{\alpha, l}\|_{L^2(\Sigma_{-r_0})} \\
&+ \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \left(\sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} + \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \right) \\
&+ \int_{-r_0}^t \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} \\
&+ \delta^{1/2} \mu_m^{-b_{|\alpha|+2}-1/2}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau \\
&+ \delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(\tau) \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u}') d\underline{u}'} d\tau.
\end{aligned} \tag{8.39}$$

9. COMMUTATOR ESTIMATES

In this section, we shall estimate the error spacetime integrals for the contributions of commutators.

Let ψ be a solution of the inhomogeneous wave equation $\square_{\tilde{g}} \psi = \rho$ and Z be a vector field, one can commute Z with the equation to derive

$$\square_{\tilde{g}}(Z\psi) = Z\rho + \frac{1}{2} \text{tr}_{\tilde{g}}^{(Z)} \tilde{\pi} \cdot \rho + c^2 \text{div}_g^{(Z)} J \tag{9.1}$$

where the vector field $^{(Z)}J$ is defined by

$$^{(Z)}J^\mu = \left(^{(Z)}\tilde{\pi}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \text{tr}_g^{(Z)} \tilde{\pi} \right) \partial_\nu \psi.$$

We remark that the raising indices for $^{(Z)}\tilde{\pi}^{\mu\nu}$ are with respect to the optic metric g .

In applications, we use the above formulas for homogeneous wave equations $\square_{\tilde{g}} \psi = 0$ and commute some commutation vector fields Z_i 's several times. Therefore, we need the following recursion formulas:

$$\begin{aligned}
\square_{\tilde{g}} \psi_n &= \rho_n, \quad \psi_n = Z\psi_{n-1}, \quad \rho_1 = 0, \\
\rho_n &= Z\rho_{n-1} + \frac{1}{2} \text{tr}_{\tilde{g}}^{(Z)} \tilde{\pi} \cdot \rho_{n-1} + c^2 \text{div}_g^{(Z)} J_{n-1}, \\
^{(Z)}J_{n-1}^\mu &= \left(^{(Z)}\tilde{\pi}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \text{tr}_g^{(Z)} \tilde{\pi} \right) \partial_\nu \psi_{n-1}.
\end{aligned} \tag{9.2}$$

Remark 9.1. When we derive energy estimates for $\square_{\tilde{g}} \psi_n = \rho_n$, due to the volume form of the conformal optic metric \tilde{g} , the integrands $\tilde{\rho}_n$ appearing in the error terms is slightly different from ρ_n . The rescaled source terms $\tilde{\rho}_n$ are defined as follows:

$$\begin{aligned}
\tilde{\rho}_n &= \frac{1}{c^2} \mu \rho_n = Z\tilde{\rho}_{n-1} + ^{(Z)}\delta \cdot \tilde{\rho}_{n-1} + ^{(Z)}\sigma_{n-1}, \\
\tilde{\rho}_1 &= 0, \quad ^{(Z)}\sigma_{n-1} = \mu \cdot \text{div}_g^{(Z)} J_{n-1}, \quad ^{(Z)}\delta = \frac{1}{2} \text{tr}_{\tilde{g}}^{(Z)} \tilde{\pi} - \mu^{-1} Z\mu + 2Z(\log(c)).
\end{aligned} \tag{9.3}$$

In view of (3.26), (3.26), (3.29) as well as the formula

$$\tilde{\text{tr}}^{(Z)} \tilde{\pi} = c \text{tr}^{(Z)} \tilde{\pi} = c \left(-\mu^{-1} ^{(Z)}\tilde{\pi}_{LL} + \text{tr}^{(Z)} \tilde{\pi} \right)$$

we have:

$$\left| ^{(T)}\delta \right| \lesssim 1, \quad \left| ^{(Q)}\delta \right| \lesssim 1, \quad \left| ^{(R_j)}\delta \right| \lesssim \delta. \tag{9.4}$$

Then error spacetime integral corresponding to $K_0 = L$ and $K_1 = \underline{L}$ containing ρ_n are as follows:

$$\begin{aligned} - \int_{W_{\underline{u}}^t} \frac{1}{c^2} \rho_n L \psi_n d\mu_g &= - \int_{W_{\underline{u}}^t} \tilde{\rho}_n L \psi_n dt d\underline{u} d\mu_g \\ - \int_{W_{\underline{u}}^t} \frac{1}{c^2} \rho_n \underline{L} \psi_n d\mu_g &= - \int_{W_{\underline{u}}^t} \tilde{\rho}_n \underline{L} \psi_n dt d\underline{u} d\mu_g \end{aligned}$$

We first consider the contribution of $^{(Z)}\sigma_{n-1}$ in $\tilde{\rho}_n$. We write $^{(Z)}\sigma_{n-1}$ in null frame $(\underline{L}, L, \frac{\partial}{\partial \theta^A})$:

$$\begin{aligned} ^{(Z)}\sigma_{n-1} &= -\frac{1}{2} L(^{(Z)}J_{n-1, \underline{L}}) - \frac{1}{2} \underline{L}(^{(Z)}J_{n-1, L}) + \text{div}(\mu^{(Z)}\not{J}_{n-1}) \\ &\quad - \frac{1}{2} \underline{L}(c^{-2}\mu)^{(Z)}J_{n-1, \underline{L}} - \frac{1}{2} \text{tr}\chi^{(Z)}J_{n-1, \underline{L}} - \frac{1}{2} \text{tr}\underline{\chi}^{(Z)}J_{n-1, L}, \end{aligned}$$

Then with the following expressions for the components of $^{(Z)}J_{n-1}$ in the null frame:

$$\begin{aligned} ^{(Z)}J_{n-1, \underline{L}} &= -\frac{1}{2} \text{tr}^{(Z)}\tilde{\not{K}}(\underline{L}\psi_{n-1}) + ^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\psi_{n-1} \\ ^{(Z)}J_{n-1, L} &= -\frac{1}{2} \text{tr}^{(Z)}\tilde{\not{K}}(L\psi_{n-1}) + ^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\psi_{n-1} - \frac{1}{2\mu} ^{(Z)}\tilde{\pi}_{LL}(\underline{L}\psi_{n-1}) \\ \mu^{(Z)}\not{J}_{n-1}^A &= -\frac{1}{2} ^{(Z)}\tilde{Z}^A(\underline{L}\psi_{n-1}) - \frac{1}{2} ^{(Z)}\tilde{\underline{Z}}^A(L\psi_{n-1}) \\ &\quad + \frac{1}{2} (^{(Z)}\tilde{\pi}_{LL} - \mu \text{tr}^{(Z)}\tilde{\not{K}}) \not{d}^A \psi_{n-1} + \mu ^{(Y)}\tilde{\not{K}}_B^A \not{d}^B \psi_{n-1} \end{aligned}$$

Based on the above expressions, we decompose:

$$^{(Z)}\sigma_{n-1} = ^{(Z)}\sigma_{1, n-1} + ^{(Z)}\sigma_{2, n-1} + ^{(Z)}\sigma_{3, n-1}$$

where $^{(Z)}\sigma_{1, n-1}$ contains the products of components of $^{(Z)}\tilde{\pi}$ with the 2nd derivatives of ψ_{n-1} , $^{(Z)}\sigma_{2, n-1}$ contains the products of the 1st derivatives of $^{(Z)}\tilde{\pi}$ with the 1st derivatives of ψ_{n-1} , and $^{(Z)}\sigma_{3, n-1}$ contains the other lower order terms. More specifically, we have:

$$\begin{aligned} ^{(Z)}\sigma_{1, n-1} &= \frac{1}{2} \text{tr}^{(Z)}\tilde{\not{K}}(\underline{L}L\psi_{n-1} + \frac{1}{2} \text{tr}_{\tilde{\not{K}}} \tilde{\chi} L\psi_{n-1}) \\ &\quad + \frac{1}{4} (\mu^{-1} ^{(Z)}\tilde{\pi}_{LL}) \underline{L}^2 \psi_{n-1} \\ &\quad - ^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\underline{L}\psi_{n-1} - ^{(Z)}\tilde{\underline{Z}} \cdot \not{d}L\psi_{n-1} \\ &\quad + \frac{1}{2} ^{(Z)}\tilde{\pi}_{LL} \not{d}^2 \psi_{n-1} + \mu ^{(Z)}\hat{\not{K}} \cdot \not{d}^2 \psi_{n-1} \end{aligned} \tag{9.5}$$

$$\begin{aligned} ^{(Z)}\sigma_{2, n-1} &= \frac{1}{4} \underline{L}(\text{tr}^{(Z)}\tilde{\not{K}}) L\psi_{n-1} + \frac{1}{4} L(\text{tr}^{(Z)}\tilde{\not{K}}) \underline{L}\psi_{n-1} \\ &\quad + \frac{1}{4} \underline{L}(\mu^{-1} ^{(Z)}\tilde{\pi}_{LL}) \underline{L}\psi_{n-1} \\ &\quad - \frac{1}{2} \not{K}_{\underline{L}} ^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\psi_{n-1} - \frac{1}{2} \not{K}_L ^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\psi_{n-1} \\ &\quad - \frac{1}{2} \text{div}^{(Z)}\tilde{\underline{Z}} \underline{L}\psi_{n-1} - \frac{1}{2} \text{div}^{(Z)}\tilde{\underline{Z}} L\psi_{n-1} \\ &\quad + \frac{1}{2} \not{d}^{(Z)}\tilde{\pi}_{LL} \not{d}\psi_{n-1} + \text{div}(\mu ^{(Z)}\hat{\not{K}}) \cdot \not{d}\psi_{n-1} \end{aligned} \tag{9.6}$$

and

$$\begin{aligned}
^{(Z)}\sigma_{3,n-1} = & \left(\frac{1}{4} \text{tr} \chi \text{tr} ^{(Z)} \tilde{\pi} + \frac{1}{4} \text{tr} \underline{\chi} (\mu^{-1} ^{(Z)} \tilde{\pi}_{LL}) \right. \\
& + \frac{1}{2} ^{(Z)} \tilde{\underline{Z}} \cdot \not{d}(c^{-2} \mu) \underline{L} \psi_{n-1} - \frac{1}{4} (\underline{L} \log(c^{-1})) \text{tr} ^{(Z)} \tilde{\pi} \underline{L} \psi_{n-1} \\
& \left. - \left(\left(\frac{1}{2} \text{tr} \chi + \underline{L}(c^{-2} \mu) \right) ^{(Z)} \tilde{\underline{Z}} + \frac{1}{2} \text{tr} \underline{\chi} ^{(Z)} \tilde{\underline{Z}} \right) \not{d} \psi_{n-1} \right)
\end{aligned} \tag{9.7}$$

With these expressions for $^{(Z)}\sigma_{n-1}$, we are able to investigate the structure of $\tilde{\rho}_n$. Basically, we want to use the recursion formulas in (9.3) to obtain a relatively explicit expression for $\tilde{\rho}_n$.

On the other hand, for the energy estimates, we consider the following possible ψ_n :

$$\psi_n = R_i^{\alpha+1} \psi, \quad \psi_n = R_i^{\alpha'} T^{l+1} \psi, \quad \psi_n = Q R_i^{\alpha'} T^l \psi$$

Here ψ_n is the n th order variation and $n = |\alpha| + 1 = |\alpha'| + l + 1$. While ψ is any first order variation. The reason that we can always first apply T , then R_i , and finally a possible Q is that the commutators $[R_i, T]$, $[R_i, Q]$ and $[T, Q]$ are one order lower than $R_i T, T R_i; Q R_i, R_i Q; Q T, T Q$ respectively. Moreover, the commutators $[R_i, T]$, $[R_i, Q]$ and $[T, Q]$ are tangent to $S_{t,\underline{u}}$. Since we let Q be the last possible commutator, there will be no Q 's in ψ_{n-1} in the second term on the right hand side of (9.5). Therefore we only need to commute Q once. Now suppose that we consider the $n = |\alpha| + 2$ th order variations of the following form:

$$\psi_{|\alpha|+2} := Z_{|\alpha|+1} \dots Z_1 \psi$$

We have the inhomogeneous wave equation:

$$\square_{\tilde{g}} \psi_{|\alpha|+2} = \rho_{|\alpha|+2}$$

As we pointed out in Remark 9.1, we define:

$$\tilde{\rho}_{|\alpha|+2} = \frac{\mu}{c^2} \rho_{|\alpha|+2}$$

Then by a induction argument, the corresponding inhomogeneous term $\tilde{\rho}_{|\alpha|+2}$ is given by:

$$\tilde{\rho}_{|\alpha|+2} = \sum_{k=0}^{|\alpha|} (Z_{|\alpha|+1} + ^{(Z_{|\alpha|+1})} \delta) \dots (Z_{|\alpha|-k+2} + ^{(Z_{|\alpha|-k+2})} \delta) ^{(Z_{|\alpha|-k+1})} \sigma_{|\alpha|-1+k} \tag{9.8}$$

9.1. Error Estimates for the lower order terms. Consider an arbitrary term in this sum. There is a total of k derivatives with respect to the commutators acting on $^{(Z)}\sigma_{|\alpha|-1+k}$. In view of the fact that $^{(Z)}\sigma_{|\alpha|-1+k}$ has the structure described in (9.5), (9.6) and (9.7), in considering the partial contribution of each term in $^{(Z)}\sigma_{1,|\alpha|-1+k}$, if the factor which is a component of $^{(Z)}\tilde{\pi}$ receives more than $\lceil \frac{|\alpha|+1}{2} \rceil$ derivatives with respect to the commutators, then the factor which is a 2nd order derivative of $\psi_{|\alpha|+1-k}$ receives at most $k - \lceil \frac{|\alpha|+1}{2} \rceil - 1$ order derivatives of commutators, thus corresponds to a derivative of the ψ of order at most: $k - \lceil \frac{|\alpha|+1}{2} \rceil + 1 + |\alpha| - k = \lceil \frac{|\alpha|}{2} \rceil + 1$, therefore this factor is bounded in $L^\infty(\Sigma_t^u)$ by the bootstrap assumption. Also, in considering the partial contribution of each term in $^{(Z)}\sigma_{2,|\alpha|+1-k}$, if the factor which is a 1st derivative of $^{(Z)}\tilde{\pi}$ receives more than $\lceil \frac{|\alpha|+1}{2} \rceil - 1$ derivatives with respect to the commutators, then the factor which is a 1st derivative of $\psi_{|\alpha|+1-k}$ receives at most $k - \lceil \frac{|\alpha|+1}{2} \rceil$ derivatives with respect to the commutators, thus corresponds to a derivative of the ψ_α of order at most $k - \lceil \frac{|\alpha|+1}{2} \rceil + 1 + |\alpha| - k = \lceil \frac{|\alpha|}{2} \rceil + 1$, therefore this factor is again bounded in $L^\infty(\Sigma_t^u)$ by the bootstrap

assumption. Similar considerations apply to $^{(Z)}\sigma_{3,|\alpha|+1-k}$. We conclude that for all the terms in the sum in (9.8) of which one factor is a derivative of the $^{(Z)}\tilde{\pi}$ of order more than $\lfloor \frac{|\alpha|+1}{2} \rfloor$, the other factor is then a derivative of the ψ_α of order at most $\lfloor \frac{|\alpha|}{2} \rfloor + 1$ and is thus bounded in $L^\infty(\Sigma_t^u)$ by the bootstrap assumption. Of these terms we shall estimate the contribution of those containing the top order spatial derivatives of the optical entities in the next subsection. Before we give the estimates for the contribution of the lower order optical terms to the spacetime integrals:

$$-\delta^{2k} \int_{W_{\underline{u}}^t} \tilde{\rho}_{\leq|\alpha|+2} L\psi_{\leq|\alpha|+2} dt d\underline{u} d\mu_g, \quad -\delta^{2k} \int_{W_{\underline{u}}^t} \tilde{\rho}_{\leq|\alpha|+2} \underline{L}\psi_{\leq|\alpha|+2} dt d\underline{u} d\mu_g, \quad (9.9)$$

we investigate the behavior of these integrals with respect to δ . Here k is the number of T s in string of commutators. For the multiplier $K_1 = \underline{L}$, the associated energy inequality is

$$\underline{E}_{\leq|\alpha|+2}(t, \underline{u}) + \underline{F}_{\leq|\alpha|+2}(t, \underline{u}) + K_{\leq|\alpha|+2}(t, \underline{u}) \lesssim \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) + \int_{W_{\underline{u}}^t} \tilde{Q}_{1, \leq|\alpha|+2}. \quad (9.10)$$

The quantities $K_{\leq|\alpha|+2}(t, \underline{u})$ are defined similar as $K(t, \underline{u})$:

$$K_{\leq|\alpha|+2}(t, \underline{u}) := \sum_{|\alpha'| \leq |\alpha|+1} \delta^{l'} K(t, \underline{u}) [Z^{\alpha'} \psi],$$

Again, l' is the number of T 's in $Z^{\alpha'}$.

In $\tilde{Q}_{1, \leq|\alpha|+2}$, there are contributions from the deformation tensors of two multipliers, which has been treated in Section 6. There are also contributions from the deformation tensors of commutators, which are given by (9.8). Now we investigate the terms which are not top order optical terms, namely, the terms containing $\underline{\chi}$ and μ of order less than $|\alpha| + 2$. In view of the discussion in Section 6, the left hand side of (9.10) is of order δ , so we expect these lower order terms in the second integral of (9.9) is of order δ . In fact, the integration on $W_{\underline{u}}^t$ gives us a δ and the multiplier $\delta^k \underline{L}\psi_{\leq|\alpha|+2}$ is of order $\delta^{1/2}$. To see the behavior of $\delta^k \sigma$, we look at σ_1 as an example. Let k' be the number of T s applied to $(\underline{L}L\psi_l + \frac{1}{2}\text{tr}\tilde{\chi}\underline{L}\psi_l)$. (4.1) implies

$$\delta^{k'} \left(\underline{L}L\psi_l + \frac{1}{2}\text{tr}\tilde{\chi}\underline{L}\psi_l \right) \sim \delta^{1/2} + \delta^{k'} \rho_l.$$

Since $\rho_1 = 0$, an induction argument implies that

$$\delta^{k'} \left(\underline{L}L\psi_l + \frac{1}{2}\text{tr}\tilde{\chi}\underline{L}\psi_l \right) \sim \delta^{1/2}.$$

Then in view of (3.24), (3.26) and (3.29), the first term in σ_1 behaves like $\delta^{1/2}$. Following the same procedure, one sees straightforwardly that all the other terms in σ_1, σ_2 and σ_3 behave like $\delta^{1/2}$ (one keeps in mind that if $Z = T$, then we multiplier a δ with the corresponding deformation tensor.) except the term $\underline{L} \left(\text{tr}^{(Z)} \tilde{\pi} \right) L\psi_l$. For this term we use the argument deriving (3.28) and Proposition 7.1 to see actually we have:

$$\|\underline{L}(\text{tr}^{(Q)} \tilde{\pi})\|_{L^\infty(\Sigma_t^u)} \lesssim \delta.$$

This completes the discussions for σ associated to K_1 .

The same argument applies to the energy inequality associated to K_0 :

$$\underline{E}_{\leq|\alpha|+2}(t, \underline{u}) + \underline{F}_{\leq|\alpha|+2}(t, \underline{u}) \lesssim \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) + \int_{W_{\underline{u}}^t} \tilde{Q}_{0, \leq|\alpha|+2}. \quad (9.11)$$

and we conclude that the lower order optical terms in the error spacetime integrals have one more power in δ than the energies on the left hand side.

Now we summarize the spacetime error estimates for the terms which come from the L^2 norms of the lower order optical quantities. In the proof of Proposition 7.4 and Proposition 7.5, we use $\mu_m^{-1/2}(t)\sqrt{E_{\leq|\alpha|+2}(t, \underline{u})}$ to control the L^2 norm $\sum_{|\alpha'|\leq|\alpha|+1} \|\not{d}Z_i^{\alpha'}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})}$. Now we just keep this L^2 norm as it is. This together with the bootstrap assumptions on the L^∞ norms of the variations implies that the contributions from the L^2 norms of lower order optical terms are bounded as:

$$\begin{aligned} & \int_{-r_0}^t \left(\sum_{|\alpha'|\leq|\alpha|+1} \delta^{1/2+l'} \|\not{d}Z_i^{\alpha'}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} + \delta \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \cdot \delta^{1/2} \int_{-r_0}^t \|\underline{L}\psi_{|\alpha|+2}\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \lesssim \int_{-r_0}^t \left(\sum_{|\alpha'|\leq|\alpha|+1} \delta^{2+2l'} \|\not{d}Z_i^{\alpha'}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 + \delta^2 E_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' + \int_0^{\underline{u}} E_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \quad \text{for } K_1 \end{aligned}$$

and

$$\begin{aligned} & \int_{-r_0}^t \left(\sum_{|\alpha'|\leq|\alpha|+1} \delta^{1/2+l'} \|\not{d}Z_i^{\alpha'}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} + \delta \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \cdot \delta^{1/2} \int_{-r_0}^t \|L\psi_{|\alpha|+2}\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \lesssim \int_{-r_0}^t \left(\sum_{|\alpha'|\leq|\alpha|+1} \delta^{1+2l'} \|\not{d}Z_i^{\alpha'}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 + \delta E_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' + \delta \int_{-r_0}^t E_{\leq|\alpha|+2}(t', \underline{u}) dt' \quad \text{for } K_0 \end{aligned}$$

where l' is the number of T 's in the string of $Z_i^{\alpha'}$. Therefore we obtain the following error estimates for the lower order optical terms:

$$\int_{-r_0}^t \delta^2 E_{\leq|\alpha|+2}(t', \underline{u}) dt' + \int_0^{\underline{u}} E_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta^2 K_{\leq|\alpha|+2}(t, \underline{u}) \quad \text{for } K_1 \quad (9.12)$$

and

$$\int_{-r_0}^t \delta E_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta K_{\leq|\alpha|+2}(t, \underline{u}) \quad \text{for } K_0 \quad (9.13)$$

Next we consider the case in which the deformation tensors receive less derivatives with respect to the commutators than the variations in the expression for $^{(Z)}\sigma_l$. More specifically, we consider the terms in the sum (9.8) in which there are at most $\lfloor \frac{|\alpha|+1}{2} \rfloor$ derivatives hitting the deformation tensor $^{(Z)}\tilde{\not{d}}$, thus the spatial derivatives on $\underline{\chi}$ is at most $\lfloor \frac{|\alpha|+1}{2} \rfloor$ and the spatial derivatives on μ is at most $\lfloor \frac{|\alpha|+1}{2} \rfloor + 1$, which are bounded in $L^\infty(\Sigma_t^{\underline{u}})$ by virtue of Proposition 7.1 and Proposition 7.2. Using the inequality $ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, we have the following estimates for these contributions:

$$\begin{aligned} & \int_{-r_0}^t \delta^2 E_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta^{-1/2} \int_0^{\underline{u}} E_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\ & + \delta^{1/2} K_{\leq|\alpha|+2}(t, \underline{u}) + \int_{-r_0}^t E_{\leq|\alpha|+2}(t', \underline{u}) dt' \quad \text{for } K_1 \end{aligned} \quad (9.14)$$

$$\int_{-r_0}^t \delta^{1/2} E_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta^{-1/2} \int_0^{\underline{u}} E_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta^{1/2} K_{\leq|\alpha|+2}(t, \underline{u}) \quad \text{for } K_0 \quad (9.15)$$

Here we estimate the terms involving $\underline{L}^2\psi_{n-1}$ and $\underline{L}\psi_{n-1}$ in terms of flux.

9.2. Top Order Optical Estimates. Now we estimate the contributions from the top order optical terms to the error spacetime integrals. In estimating the top order optical terms, we need to choose the power of $\mu_m(t)$ large enough. Therefore from this subsection on, we will use C to denote an absolute positive constant so that one can see the largeness of the power of $\mu_m(t)$ more clearly.

The top order optical terms come from the term in which all the commutators hit the deformation tensors in the expression of $^{(Z_1)}\sigma_1$, namely, the term:

$$(Z_{|\alpha|+1} + {}^{(Z_{|\alpha|+1})}\delta) \dots (Z_2 + {}^{(Z_2)}\delta) {}^{(Z_1)}\sigma_1$$

more precisely, in:

$$Z_{|\alpha|+1} \dots Z_2 \left(-\frac{1}{2} L({}^{(Z_1)}J_{1,\underline{L}}) - \frac{1}{2} \underline{L}({}^{(Z_1)}J_{1,L}) + \text{div}(\mu^{(Z_1)}\sharp) \right)$$

when the operators $L, \underline{L}, \text{div}$ hit the deformation tensors in the expression of $^{(Z_1)}J$.

Now we consider the top order variations:

$$R_i^{\alpha+1}\psi, \quad R_i^{\alpha'} T^{l+1}\psi, \quad QR_i^{\alpha'} T^l\psi$$

where $|\alpha| = N_{\text{top}} - 1$ and $|\alpha'| + l + 1 = |\alpha| + 1$. Then the corresponding principal optical terms are:

$$\begin{aligned} \tilde{\rho}_{|\alpha|+2}(R_i^{\alpha+1}\psi) &:= \frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi, \quad \tilde{\rho}_{|\alpha|+2}(R_i^{\alpha'} T^{l+1}\psi) := \frac{1}{c}(R_i^{\alpha'} T^l \underline{\Delta}\mu) \cdot T\psi \\ \tilde{\rho}_{|\alpha|+2}(QR_i^{\alpha'} T^l\psi) &:= \frac{t\mu}{c}(\sharp R_i^{\alpha'} \text{tr}\underline{\chi}') \cdot \sharp\psi + \frac{t\mu}{c}(\sharp R_i^{\alpha'} \underline{\Delta}\mu) \cdot \underline{L}\psi, \quad \text{if } l = 0 \\ \tilde{\rho}_{|\alpha|+2}(QR_i^{\alpha'} T^l\psi) &:= \frac{t\mu}{c}(\sharp R_i^{\alpha'} T^{l-1} \underline{\Delta}\mu) \cdot \sharp\psi + \frac{t\mu}{c}(\sharp R_i^{\alpha'} T^l \underline{\Delta}\mu) \cdot \underline{L}\psi, \quad \text{if } l \geq 1 \end{aligned}$$

Here we used the structure equation (2.34)

$$\sharp_T \text{tr}\underline{\chi} = \underline{\Delta}\mu + \mathcal{O}_{\geq 0}^{\leq 1}$$

Now we briefly investigate the behavior of the above terms with respect to δ . Note that $R_i^\alpha Q\psi$ has the same behavior as $R_i^{\alpha+1}\psi$ with respect to δ , while for their corresponding top order optical terms $\frac{t\mu}{c}(\sharp R_i^\alpha \text{tr}\underline{\chi}') \cdot \sharp\psi$ and $\frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi$, the former behaves better than the latter with respect to δ :

$$|\frac{t\mu}{c}(\sharp R_i^\alpha \text{tr}\underline{\chi}') \cdot \sharp\psi| \sim |\mu(R_i^{\alpha+1}\text{tr}\underline{\chi}')| \delta^{1/2}, \quad |\frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi| \sim |(R_i^{\alpha+1}\text{tr}\underline{\chi}')| \delta^{-1/2}$$

We see that not only the former behaves better with respect to δ , but also has an extra μ , which makes the behavior even better when μ is small. This means that we only need to estimate the contribution of $\frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi$. The same analysis applies to the terms involving $\underline{L}\psi$ as well as the comparison between $R_i^{\alpha'} T\psi$ and $QR_i^{\alpha'} T\psi$, which correspond to $\frac{1}{c}(R_i^{\alpha'} \underline{\Delta}\mu) \cdot T\psi$ and $\frac{t\mu}{c}(R_i^{\alpha'+1} \underline{\Delta}\mu) \cdot \sharp\psi$. So in the following, we do not need to estimate the contributions corresponding to the variations containing a Q .

9.2.1. Contribution of K_0 . In this subsection we first estimate the spacetime integral:

$$\begin{aligned} &\int_{W_{\underline{u}}^t} \frac{1}{c} |R_i^{\alpha+1}\text{tr}\underline{\chi}'| |T\psi| |LR_i^{\alpha+1}\psi| dt' du' d\mu_{\sharp} \lesssim \\ &\int_{-r_0}^t \sup_{\Sigma_{t'}^{\underline{u}}} (\mu^{-1} |T\psi|) \|\mu \sharp R_i^{\alpha} \text{tr}\underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \end{aligned} \tag{9.16}$$

By (8.23), we have:

$$\begin{aligned} \|\mu \not{d} R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_t^u)} &\lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ &\quad + \int_{-r_0}^t \delta^{1/2} \mu_m^{-b_{|\alpha|+2}-1/2}(\tau) \sqrt{\tilde{E}_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau \\ &\quad + \delta^{-1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\int_0^u \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}'}. \end{aligned}$$

By the monotonicity of $\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})$ in t , we have

$$\begin{aligned} \|\mu \not{d} R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_t^u)} &\lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ &\quad + \delta^{1/2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \\ &\quad + \delta^{-1/2} \sqrt{\int_0^u \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}'} \mu_m^{-b_{|\alpha|+2}}(t) \end{aligned} \quad (9.17)$$

Without loss of generality, here we assume that there is a $t_0 \in [-r_0, t^*)$ such that $\mu_m(t_0) = \frac{1}{10}$ and $\mu_m(t') \geq \frac{1}{10}$ for $t \leq t_0$. If there is no such t_0 in $[-r_0, t^*)$, then $\mu_m(t)$ has an absolute positive lower bound for all $[-r_0, t^*)$ and it is clear to see that the following argument simplifies and also works in this case. In view of part (1') and part (2) in Lemma 8.1 and the fact $\mu_m(t') \geq \frac{1}{10}$ for $t' \in [-r_0, t_0]$, we have

$$\begin{aligned} \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' &\lesssim \mu_m^{-b_{|\alpha|+2}+1/2}(t_0) \leq \mu_m^{-b_{|\alpha|+2}+1/2}(t), \\ \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' &\lesssim \frac{1}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-b_{|\alpha|+2}+1/2}(t). \end{aligned}$$

Therefore the second term in (9.17) are bounded by:

$$\delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}$$

Substituting this in (9.16), and using the fact that $|T\psi| \lesssim \delta^{-1/2}$, we see that the spacetime integral (9.16) is bounded by:

$$\begin{aligned} &\int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^u)} dt' \\ &+ \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^u)} dt' \\ &+ \int_{-r_0}^t \delta^{-1} \mu_m^{-b_{|\alpha|+2}-1}(t') \sqrt{\int_0^u \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}') d\underline{u}'} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^u)} dt' \end{aligned}$$

For the factor $\|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_t^u)}$, we bound it as:

$$\|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_t^u)} \leq \sqrt{E_{|\alpha|+2}(t, \underline{u})} \leq \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}$$

Then the spacetime integral (9.16) is bounded by:

$$\begin{aligned} & \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\ & + \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ & + \delta^{-1} \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}'} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \end{aligned}$$

Splitting the above integrals as $\int_{-r_0}^{t_0} + \int_{t_0}^t$, we see that the “non-shock” part $\int_{-r_0}^{t_0}$ is bounded by:

$$\int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') \left(\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \delta^{-2} \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt' \quad (9.18)$$

using part (2) of Lemma 8.1.

$$\begin{aligned} & \boxed{\frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \frac{C}{(2b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}+1/2}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ & + \delta^{-2} \frac{C}{(2b_{|\alpha|+2})} \mu_m^{-2b_{|\alpha|+2}}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \end{aligned} \quad (9.19)$$

using part (1') of Lemma 8.1.

Remark 9.2. The boxed term is from the estimates for the top order term F_α . In view of (8.3), the number of top order terms contributed by the variations is independent of δ and $|\alpha|$, so is the constant C in the boxed term. Later on in the top order energy estimates we will choose $b_{|\alpha|+2}$ in such a way that $\frac{C}{b_{top}} \leq \frac{1}{10}$. (The purpose of doing this is to make sure that this term can be absorbed by the left hand side the energy inequality.) Therefore we can choose $b_{top} = \{10, \frac{C}{20}\}$. In particular b_{top} is independent of δ .

Next we consider the spacetime integral:

$$\begin{aligned} & \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{1}{c} |R_i^{\alpha'} T^l \Delta \mu| |T\psi| |LR^{\alpha'} T^{l+1} \psi| dt' d\underline{u}' d\mu_\sharp \lesssim \\ & \delta^{2l+2} \int_{-r_0}^t \sup_{\Sigma_{t'}^{\underline{u}}} (\mu^{-1} |T\psi|) \|\mu R_i^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LR^{\alpha'} T^{l+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \end{aligned} \quad (9.20)$$

By (8.39) and the monotonicity of $\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})$, $\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})$ and $\tilde{F}_{\leq|\alpha|+2}(t, \underline{u})$ in t , we have:

$$\begin{aligned} & \delta^{l+1} \|\mu R_i^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \leq C \delta^{l+1} \|F_{\alpha, l}\|_{L^2(\Sigma_{-r_0}^{\underline{u}})} \\ & + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ & + \int_{-r_0}^t \left(\delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right. \\ & + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \Big) dt' \\ & + \delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}'} dt' \end{aligned} \quad (9.21)$$

As before, we have:

$$\int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t') dt' \leq C \mu_m^{-b_{|\alpha|+2}+1}(t), \quad \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \leq C \mu_m^{-b_{|\alpha|+2}+1/2}(t)$$

Therefore

$$\begin{aligned} \delta^{l+1} \|\mu R_i^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_{t'}^u)} &\leq C \delta^{l+1} \|F_{\alpha,l}\|_{L^2(\Sigma_{-r_0}^u)} \\ &\quad + C \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + C \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + C \delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1}(t) \sqrt{\int_0^u \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}'} \end{aligned} \quad (9.22)$$

Again, we estimate the other factor $\|LR_i^{\alpha'} T^{l+1} \psi\|_{L^2(\Sigma_t^u)}$ by:

$$\delta^{l+1} \|LR_i^{\alpha'} T^{l+1} \psi\|_{L^2(\Sigma_t^u)} \leq \sqrt{E_{\leq|\alpha|+2}(t, \underline{u})} \leq \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}$$

Here we also split the spacetime integral (9.20) into "shock part $\int_{t_0}^t$ " and "non-shock part $\int_{-r_0}^{t_0}$ ". We first estimate the "shock part". Due to the estimate $|T\psi| \lesssim \delta^{-1/2}$, the contribution of the first term in (9.21) to the spacetime integral (9.20) is bounded by:

$$\begin{aligned} &\int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \|LR_i^{\alpha'} T^{l+1} \psi\|_{L^2(\Sigma_{t'}^u)} dt' \\ &\leq C \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \leq C \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') dt' \\ &\leq \frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned}$$

Therefore, the contributions from the first two terms in (9.21) are bounded by:

$$\frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) [\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})]$$

and the contributions from the rest terms in (9.21) are bounded by:

$$\begin{aligned} &\frac{C}{(2b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}+1/2}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ &\quad + \frac{C}{(2b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) [\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \int_0^u \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}'] \end{aligned}$$

Therefore "shock-part" of the spacetime integral (9.20) is bounded by:

$$\frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) [\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})] + \frac{C}{(2b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \int_0^u \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \quad (9.23)$$

As before, for the "non-shock part", which mean $t' \in [-r_0, t_0]$, we have a positive lower bound for $\mu_m(t')$, then the spacetime integral (9.20) is bounded by:

$$\int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') [\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \int_0^u \tilde{F}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}'] dt' \quad (9.24)$$

which will be treated by Gronwall.

Finally, using the inequality $ab \leq \frac{1}{2}b^2 + \frac{1}{2}b^2$, the initial contribution $\|F_{\alpha,l}\|_{L^2(\Sigma_{-r_0})}$ is bounded by

$$\delta^{2l+2} \|F_{\alpha,l}\|_{L^2(\Sigma_{-r_0})}^2 + \int_{-r_0}^t \mu_m^{-2b|\alpha|+2-1}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt'. \quad (9.25)$$

The second term above has already been estimated.

9.2.2. *Contribution of K_1 .* In this subsection we estimate the contributions of top order optical terms associated to K_1 . We start with the following absolute value of a spacetime integral:

$$\left| \int_{W_{\underline{u}}^t} \frac{1}{c} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (\underline{L} R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \right| \quad (9.26)$$

Since

$$\tilde{g} = \frac{1}{c} g, \quad d\mu_{\tilde{g}} = \frac{1}{c} d\mu_g$$

the above spacetime integral can be written as:

$$\int_{W_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (\underline{L} R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \quad (9.27)$$

Let us set:

$$F(t, \underline{u}) = \int_{S_{t, \underline{u}}} f d\mu_{\tilde{g}}$$

then

$$\frac{\partial F}{\partial t} = \int_{S_{t, \underline{u}}} (\underline{L} f + \tilde{\text{tr}} \underline{\chi} f) d\mu_{\tilde{g}}$$

So we may convert the spacetime integral into two hypersurface integrals:

$$\int_{W_{\underline{u}}^t} (\underline{L} f + \tilde{\text{tr}} \underline{\chi} f) d\mu_{\tilde{g}} d\underline{u}' dt' = \int_{\Sigma_t^{\underline{u}}} f d\mu_{\tilde{g}} d\underline{u}' - \int_{\Sigma_{-r_0}^{\underline{u}}} f d\mu_{\tilde{g}} d\underline{u}'$$

So we write the spacetime integral (9.27) as:

$$\begin{aligned} & \int_{W_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (\underline{L} R_i^{\alpha+1} \psi + \tilde{\text{tr}} \underline{\chi} R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & - \int_{W_{\underline{u}}^t} \tilde{\text{tr}} \underline{\chi} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & = \int_{W_{\underline{u}}^t} (\underline{L} + \tilde{\text{tr}} \underline{\chi}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (R_i^{\alpha+1} \psi) \right] dt' d\underline{u}' d\mu_{\tilde{g}} \\ & - \int_{W_{\underline{u}}^t} (\underline{L} + \tilde{\text{tr}} \underline{\chi}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \right] \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \end{aligned}$$

We conclude that (9.27) equals:

$$\begin{aligned} & \int_{\Sigma_t^{\underline{u}}} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (R_i^{\alpha+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\ & - \int_{\Sigma_{-r_0}^{\underline{u}}} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (R_i^{\alpha+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\ & - \int_{W_{\underline{u}}^t} (\underline{L} + \tilde{\text{tr}} \underline{\chi}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \right] \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \end{aligned}$$

We first consider the hypersurface integral which, integrating by parts, equals:

$$-H_0 - H_1 - H_2$$

where:

$$\begin{aligned} H_0 &= \int_{\Sigma_t^{\underline{u}}} (R_i^\alpha \text{tr} \underline{\chi}) \cdot (T\psi) \cdot (R_i^{\alpha+2} \psi) d\underline{u}' d\mu_{\tilde{g}} \\ H_1 &= \int_{\Sigma_t^{\underline{u}}} (R_i^\alpha \text{tr} \underline{\chi}') \cdot (R_i T\psi) \cdot (R_i^{\alpha+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\ H_2 &= \int_{\Sigma_t^{\underline{u}}} (T\psi) \cdot (R_i^{\alpha+1} \psi) \cdot (R^\alpha \text{tr} \underline{\chi}') \left(\frac{1}{2} \text{tr}^{(R_i)} \tilde{\pi} \right) d\underline{u}' d\mu_{\tilde{g}} \end{aligned}$$

Since we shall bound both $T\psi$ and $R_i T\psi$ in L^∞ norm, compared to H_0 , H_1 is a lower order term with respect to the order of derivatives. While for H_2 , we use the estimate:

$$|\text{tr}^{(R_i)} \tilde{\pi}| \leq C\delta$$

to see that it is a lower order term with respect to both the behavior of δ and the order of derivatives compared to H_0 . This analysis tells us that we only need to estimate H_0 .

$$\begin{aligned} |H_0| &\leq \int_{\Sigma_t^{\underline{u}}} |T\psi| |R_i^\alpha \text{tr} \underline{\chi}'| |\not{d} R_i^{\alpha+1} \psi| d\underline{u}' d\mu_{\tilde{g}} \leq C\delta^{-1/2} \|R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_t^{\underline{u}})} \|\not{d} R_i^{\alpha+1} \psi\|_{L^2(\Sigma_t^{\underline{u}})} \\ &\leq C \int_{-r_0}^t \left(\mu_m^{-1/2}(t') \sqrt{\underline{E}_{|\alpha|+2}(t', \underline{u})} \right) dt' \cdot \mu_m^{-1/2}(t) \sqrt{\underline{E}_{|\alpha|+2}(t, \underline{u})} \end{aligned}$$

by Proposition 7.4. Then in terms of modified energies, we have:

$$|H_0| \leq C \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{\underline{E}}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u})}$$

Again, we consider the “shock part $\int_{-r_0}^{t_0}$ ” and “non-shock part $\int_{t_0}^t$ ”, which are denoted by H_0^S and H_0^N , separately.

In the “shock part”, by the monotonicity of $\tilde{\underline{E}}_{\leq|\alpha|+2}(t)$, we have:

$$\begin{aligned} |H_0^S| &\leq C \sqrt{\tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u})} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\leq C \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) \mu_m^{-b_{|\alpha|+2}-1/2}(t) \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \\ &\leq \frac{C}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) \cdot \mu_m^{-b_{|\alpha|+2}+1/2}(t) \\ &= \frac{C}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned}$$

The estimates for the “non-shock part” is more delicate.

$$|H_0^N| \leq C \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{\underline{E}}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u})}$$

Now for $t' \in [-r_0, t_0]$, $\mu_m^{-1}(t') \leq 10$. This means we have the following estimate for the above integral:

$$\begin{aligned} \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' &= \int_{-r_0}^{t_0} \mu_m^{-1}(t') \mu_m^{-b_{|\alpha|+2}+1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\leq C \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}+1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \leq C \int_{-r_0}^{t_0} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-b_{|\alpha|+2}+1/2}(t) \end{aligned}$$

where in the last step, we have used part (2) of Lemma 8.1.

Therefore

$$\begin{aligned} |H_0^N| &\leq C \int_{-r_0}^{t_0} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-2b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\leq \epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + C_\epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \end{aligned}$$

Here ϵ is a small absolute constant to be determined later.

We obtain the following estimate for $|H_0|$:

$$\begin{aligned} |H_0| &\leq \frac{C}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ &\quad + C_\epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned} \tag{9.28}$$

Next we consider the spacetime integral:

$$\begin{aligned} &\int_{W_{\underline{u}}^t} (\underline{L} + \tilde{\text{tr}}\underline{\chi}) \left[(R_i^{\alpha+1} \text{tr}\underline{\chi}) \cdot (T\psi) \right] \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ &= \int_{W_{\underline{u}}^t} ((\underline{L} + \tilde{\text{tr}}\underline{\chi})(R_i^{\alpha+1} \text{tr}\underline{\chi}')) \cdot (T\psi) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ &\quad + \int_{W_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr}\underline{\chi}') \cdot \underline{L}(T\psi) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} := I + II \end{aligned}$$

For the second term in the above sum, by the fact that:

$$|\underline{L}(T\psi)| \leq C\delta^{-1/2}$$

we have:

$$|II| \leq C\delta^{-1/2} \int_{-r_0}^t \|\underline{L} R_i^\alpha \text{tr}\underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|R_i^{\alpha+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'$$

On the other hand, by Lemma 7.3, we have:

$$\|R_i^{\alpha+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} \leq C\delta (E_{|\alpha|+2}(t', \underline{u}) + \underline{E}_{|\alpha|+2}(t', \underline{u}))$$

which gives:

$$|II| \leq C\delta^{1/2} \int_{-r_0}^t \|\underline{L} R_i^\alpha \text{tr}\underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} (E_{|\alpha|+2}(t', \underline{u}) + \underline{E}_{|\alpha|+2}(t', \underline{u})) dt'$$

This has a similar form as (9.16), and moreover, it has an extra δ , which is consistent with the order of $\underline{E}_{|\alpha|+2}(t, \underline{u})$. (We already took into account the effect of $\underline{L}(T\psi)$.) So this term is already handled.

To estimate $|I|$, we first rewrite:

$$(\underline{L} + \tilde{\text{tr}}\underline{\chi})(R_i^{\alpha+1} \text{tr}\underline{\chi}') = R_i(\underline{L} + \tilde{\text{tr}}\underline{\chi})(R_i^\alpha \text{tr}\underline{\chi}') + {}^{(R_i)}\underline{Z} R_i^\alpha \text{tr}\underline{\chi}' - R_i(\text{tr}\underline{\chi}') R_i^\alpha \text{tr}\underline{\chi}' + \text{l.o.t.}$$

The contribution of the second term is:

$$\int_{W_{\underline{u}}^t} |T\psi|^{(R_i)} \underline{Z} \|\not{d} R_i^\alpha \text{tr} \underline{\chi}'\| R_i^{\alpha+1} \psi | dt' d\underline{u}' d\mu_{\tilde{g}}$$

By the estimates:

$$|^{(R_i)} \underline{Z}| \leq C\delta, \quad |T\psi| \leq C\delta^{-1/2}$$

this contribution is bounded by:

$$\delta^{1/2} \int_{-r_0}^t \|\not{d} R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|R_i^{\alpha+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'$$

which is similar to the estimate for $|II|$ and has an extra δ , so this is a lower order term.

By the estimate

$$|R_i \text{tr} \underline{\chi}| \leq C\delta,$$

the contribution of the third term is bounded by:

$$C\delta \int_{W_{\underline{u}}^t} |R_i^\alpha \text{tr} \underline{\chi}'| |T\psi| |R_i^{\alpha+1} \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \leq C\delta^{1/2} \int_{-r_0}^t \|R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|R_i^{\alpha+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'$$

By proposition 7.4 and Lemma 7.3, this is bounded in terms of modified energies by:

$$\begin{aligned} & \delta^2 \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \cdot \mu_m^{-b_{|\alpha|+2}}(t') (\sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})}) dt' \quad (9.29) \\ & \leq C\delta^2 (\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})) \int_{-r_0}^t \mu_m^{-2a-1/2}(t') dt' \\ & \leq C\delta^2 \mu_m^{-2b_{|\alpha|+2}+1/2}(t) (\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})) \end{aligned}$$

Again, we used Lemma 8.1 in the last step.

Now we are left with the spacetime integral:

$$\int_{W_{\underline{u}}^t} \left(R_i(\underline{L} + \tilde{\text{tr}} \underline{\chi})(R_i^\alpha \text{tr} \underline{\chi}') \right) \cdot (T\psi) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}}$$

Integrating by parts, this equals:

$$\begin{aligned} & - \int_{W_{\underline{u}}^t} \left((\underline{L} + \tilde{\text{tr}} \underline{\chi})(R_i^\alpha \text{tr} \underline{\chi}') \right) \cdot (T\psi) \cdot (R_i^{\alpha+2} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & - \int_{W_{\underline{u}}^t} \left((\underline{L} + \tilde{\text{tr}} \underline{\chi})(R_i^\alpha \text{tr} \underline{\chi}') \right) \cdot (R_i T\psi + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\not{d}}) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \quad \quad \quad := -V_1 - V_2 \end{aligned}$$

By the estimate:

$$|\text{tr}^{(R_i)} \tilde{\not{d}}| \leq C\delta$$

we see that:

$$|R_i T\psi + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\not{d}}| \leq C\delta^{-1/2}, \quad |T\psi| \leq C\delta^{-1/2}$$

So compared to V_1 , V_2 is a lower order term, and we use Lemma 7.3 to estimate:

$$\|R_i^{\alpha+1} \psi\|_{L^2(\Sigma_t^{\underline{u}})} \leq C\delta \left(\sqrt{E_{|\alpha|+2}(t, \underline{u})} + \sqrt{\underline{E}_{|\alpha|+2}(t, \underline{u})} \right)$$

So we only need to estimate $|V_1|$, which is bounded by:

$$\begin{aligned} & \int_{W_{\underline{u}}^t} \left| \left(\underline{L} + \frac{2}{t - \underline{u}} \right) (R_i^\alpha \text{tr} \underline{\chi}') \right| |T\psi| |R_i^{\alpha+2} \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & + \int_{W_{\underline{u}}^t} |\tilde{\text{tr}} \underline{\chi}'| |R_i^\alpha \text{tr} \underline{\chi}'| |T\psi| |R_i^{\alpha+2} \psi| dt' d\underline{u}' d\mu_{\tilde{g}} := V_{11} + V_{12} \end{aligned}$$

By the estimates:

$$|\tilde{\text{tr}} \underline{\chi}'| \leq C\delta, \quad |T\psi| \leq C\delta^{-1/2}$$

and proposition 7.4, V_{12} is bounded in terms of modified energies by:

$$\begin{aligned} & \delta^{1/2} \int_{-r_0}^t \|R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|\underline{\Delta} R_i^{\alpha+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \leq C\delta \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ & \leq C\delta \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') dt' \leq C\delta \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}). \end{aligned} \tag{9.30}$$

In the last step we used Lemma 8.1.

To estimate V_{11} , we recall the propagation equation for $\text{tr} \underline{\chi}'$:

$$\underline{L} \text{tr} \underline{\chi}' + \frac{2}{t - \underline{u}} \text{tr} \underline{\chi}' = e \text{tr} \underline{\chi}' - |\underline{\chi}'|^2 + \frac{2e}{t - \underline{u}} - \text{tr} \underline{\alpha}' := \rho_0$$

Applying R_i^α to this equation gives:

$$\underline{L} R_i^\alpha \text{tr} \underline{\chi}' + \frac{2}{t - \underline{u}} R_i^\alpha \text{tr} \underline{\chi}' = \sum_{|\beta| \leq |\alpha|} R_i^{\beta(R_i)} \underline{L} R_i^{\alpha-\beta-1} \text{tr} \underline{\chi}' + R_i^\alpha \rho_0$$

Again, by the estimates

$$|^{(R_i)} \underline{L}| \leq C\delta, \quad |T\psi| \leq C\delta^{-1/2}$$

as well as Proposition 7.4, the contribution of the first term on the right hand side is bounded in terms of modified energies by:

$$\begin{aligned} & \delta^{1/2} \int_{-r_0}^t \|R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|R_i^{\alpha+2} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \leq C\delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \leq C\delta \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}). \end{aligned} \tag{9.31}$$

The following terms in $R_i^\alpha \rho_0$ also enjoy the above estimates:

$$R_i^\alpha (|\underline{\chi}'|^2), \quad (R_i^\beta e) \cdot (R_i^{\alpha-\beta} \text{tr} \underline{\chi}') \quad \text{where } |\beta| \leq |\alpha| - |\beta|$$

While the other contributions from $R_i^\alpha e \cdot \text{tr} \underline{\chi}'$, $\frac{2e}{t - \underline{u}}$ and the lower order terms in $\text{tr} \underline{\alpha}'$ can be bounded by:

$$C \mu_m^{-2b_{|\alpha|+2}+1/2}(t) \left(\delta^{3/2} \tilde{E}_{|\alpha|+2}(t, \underline{u}) + \delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \right) \tag{9.32}$$

in view of Lemma 7.3.

Now we estimate the contribution from the principal term in $\text{tr} \underline{\alpha}'$, which is:

$$\frac{dc^2}{d\rho} \psi_0 \underline{\Delta} R_i^\alpha \psi_0$$

In view of the estimates:

$$|\psi_0| \leq C\delta^{1/2}, \quad |T\psi| \leq C\delta^{-1/2}$$

this contribution is bounded by:

$$\begin{aligned} \int_{-r_0}^t \|\not{d}R_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|R_i^{\alpha+2}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' &\leq C \int_{-r_0}^t \|\not{d}R_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \\ &\leq C \int_{-r_0}^t \mu_m^{-2b|\alpha|+2-1}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \end{aligned}$$

Again, considering the “shock part” and “non-shock part” in regard to this integral, we obtain that it is bounded by:

$$\frac{C}{2b|\alpha|+2} \mu_m^{-2b|\alpha|+2}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \int_{-r_0}^{t_0} \mu_m^{-2b|\alpha|+2}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \quad (9.33)$$

This completes the estimates for the spacetime integral (9.26).

Next we consider the top order optical contribution of the variation $R^{\alpha'} T^{l+1}\psi$, where $|\alpha'|+l+1 = |\alpha|+1$, which is the following spacetime integral:

$$\delta^{2l+2} \left| \int_{W_{\underline{u}}^t} (T\psi) \cdot (R_i^{\alpha'} T^l \not{d}\mu) \cdot (\underline{L} R_i^{\alpha'} T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \right| \quad (9.34)$$

Again, we rewrite the above spacetime integral as:

$$\begin{aligned} &\delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \cdot (R_i^{\alpha'} T^l \not{d}\mu) \cdot ((\underline{L} + \tilde{\text{tr}}\chi)(R_i^{\alpha'} T^{l+1}\psi)) dt' d\underline{u}' d\mu_{\tilde{g}} \\ &- \delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \cdot (R_i^{\alpha'} T^l \not{d}\mu) \cdot (\tilde{\text{tr}}\chi(R_i^{\alpha'} T^{l+1}\psi)) dt' d\underline{u}' d\mu_{\tilde{g}} \end{aligned}$$

which is:

$$\begin{aligned} &\delta^{2l+2} \int_{W_{\underline{u}}^t} (\underline{L} + \tilde{\text{tr}}\chi) \left((T\psi)(R_i^{\alpha'} T^l \not{d}\mu)(R_i^{\alpha'} T^{l+1}\psi) \right) dt' d\underline{u}' d\mu_{\tilde{g}} \\ &- \delta^{2l+2} \int_{W_{\underline{u}}^t} (\underline{L} T\psi)(R_i^{\alpha'} T^l \not{d}\mu)(R_i^{\alpha'} T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ &- \delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi)((\underline{L} + \tilde{\text{tr}}\chi)(R_i^{\alpha'} T^l \not{d}\mu))(R_i^{\alpha'} T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \end{aligned}$$

As before, the spacetime integral in the first line above can be written as:

$$\delta^{2l+2} \int_{\Sigma_{\underline{u}}^t} (T\psi)(R_i^{\alpha'} T^l \not{d}\mu)(R_i^{\alpha'} T^{l+1}\psi) d\underline{u}' d\mu_{\tilde{g}} - \delta^{2l+2} \int_{\Sigma_{-r_0}^{\underline{u}}} (T\psi)(R_i^{\alpha'} T^l \not{d}\mu)(R_i^{\alpha'} T^{l+1}\psi) d\underline{u}' d\mu_{\tilde{g}}$$

We shall only estimate the integral on $\Sigma_{\underline{u}}^t$, which can be written as:

$$\begin{aligned} &- \delta^{2l+2} \int_{\Sigma_{\underline{u}}^t} (T\psi)(R_i^{\alpha'-1} T^l \not{d}\mu)(R_i^{\alpha'+1} T^{l+1}\psi) d\underline{u}' d\mu_{\tilde{g}} \\ &- \delta^{2l+2} \int_{\Sigma_{\underline{u}}^t} \left((R_i T\psi) + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\pi} \right) (R_i^{\alpha'-1} T^l \not{d}\mu)(R_i^{\alpha'+1} T^{l+1}\psi) d\underline{u}' d\mu_{\tilde{g}} \\ &:= -H'_0 - H'_1 \end{aligned}$$

By the estimates:

$$|R_i T \psi| \leq C \delta^{-1/2}, \quad |\text{tr}^{(R_i)} \tilde{\pi}| \leq C \delta$$

we see that compared to H'_0 , H'_1 is a lower order term, so here we only give the estimates for H'_0 :

$$|H'_0| \leq C \delta^{-1/2+2l+2} \|R^{\alpha'-1} T^l \Delta \mu\|_{L^2(\Sigma_t^u)} \|\mathbb{A} R_i^{\alpha'} T^{l+1} \psi\|_{L^2(\Sigma_t^u)}$$

In view of Proposition 7.5, this is bounded in terms of modified energies by:

$$\begin{aligned} |H'_0| &\leq C \int_{-r_0}^t \delta \left(\mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\ &\quad \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \end{aligned}$$

Again, we need to consider the “shock part $H_0'^S$ ” and the “non-shock part $H_0'^N$ ” separately.

For $H_0'^S$, we have:

$$\begin{aligned} |H_0'^S| &\leq C \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad \delta \left(\sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' + \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}}(t') dt' \right) \\ &\leq \frac{C \delta}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ &\quad + \frac{C \delta}{(b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \end{aligned}$$

For $H_0'^N$, we use the same argument as we did in estimating (9.27). Since $\mu_m^{-1}(t') \leq C$ for $t' \in [-r_0, t_0]$ with an absolute constant C , we have, by 8.10:

$$\begin{aligned} \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' &\leq C \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}+1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\leq C \int_{-r_0}^{t_0} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-b_{|\alpha|+2}+1/2}(t) \end{aligned}$$

The same argument applies to the integral involving $\delta \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}$, so finally we obtain:

$$\begin{aligned} |H_0'^N| &\leq C \delta \int_{-r_0}^{t_0} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-2b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + C \int_{-r_0}^{t_0} \delta \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-2b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\leq C \delta \mu_m^{-2b_{|\alpha|+2}}(t) \left(\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \sqrt{E_{\leq|\alpha|+2}(t, \underline{u})} \right) \end{aligned}$$

We finally obtain the estimates for $|H'_0|$:

$$|H'_0| \leq C \delta \mu_m^{-2b_{|\alpha|+2}}(t) \left(\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \sqrt{E_{\leq|\alpha|+2}(t, \underline{u})} \right) \quad (9.35)$$

Finally, we estimate the spacetime integrals:

$$\begin{aligned}
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} (\underline{L}T\psi)(R_i^{\alpha'} T^l \Delta\mu)(R_i^{\alpha'} T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \left((\underline{L} + \frac{2}{t-\underline{u}}) (R_i^{\alpha'} T^l \Delta\mu) \right) (R_i^{\alpha'} T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \cdot \widetilde{\text{tr}}\underline{\chi}'(R_i^{\alpha'} T^l \Delta\mu) \cdot (R_i^{\alpha'} T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& := -V_1' - V_2' - V_3'
\end{aligned}$$

By the estimates:

$$|T\psi| \leq C\delta^{-1/2}, \quad |\underline{L}T\psi| \leq C\delta^{-1/2}, \quad |\widetilde{\text{tr}}\underline{\chi}'| \leq C\delta$$

we see that, compared to V_1' , V_3' is a lower order term. By Lemma 7.3 and (9.22), V_1' is bounded by:

$$\begin{aligned}
|V_1'| & \leq C\delta^{-1/2} \int_{-r_0}^t \sup_{\Sigma_{t'}^{\underline{u}}} (\mu^{-1}) \delta^{l+1} \|\mu R_i^{\alpha'} T^l \Delta\mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \\
& \cdot \delta \left(\mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\
& \leq C \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') \left(\sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \right) \\
& \cdot \delta \left(\sqrt{\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\
& + C \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}}(t') \sqrt{\int_0^{\underline{u}} \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}'} \left(\sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \right) dt'
\end{aligned}$$

Again, we consider the “shock part” and “non-shock part” separately. When $t' \in [-r_0, t_0]$, since $\mu_m^{-1}(t') \leq C$, the above integrals are bounded by:

$$\begin{aligned}
& C\delta \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') (\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})) dt' \\
& + C \int_{-r_0}^{t_0} \delta^{-1} \mu_m^{-2b_{|\alpha|+2}}(t') \left(\int_0^{\underline{u}} \widetilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt'
\end{aligned}$$

While when $t' \in [t_0, t]$, we have the following estimates:

$$\begin{aligned}
& C\delta (\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})) \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') dt' \\
& + \delta^{-1} \left(\int_0^{\underline{u}} \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right) \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2}}(t') dt' \\
& \leq \frac{C\delta}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) (\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})) \\
& + \frac{C\delta^{-1}}{(2b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \int_0^{\underline{u}} \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}'
\end{aligned}$$

Therefore V'_1 is bounded by:

$$\begin{aligned}
& \frac{C\delta}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) (\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})) \\
& + \frac{C\delta^{-1}}{(2b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + C\delta \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') (\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u})) dt' \\
& + C \int_{-r_0}^{t_0} \delta^{-1} \mu_m^{-2b_{|\alpha|+2}}(t') \left(\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt'
\end{aligned} \tag{9.36}$$

Now we estimate V'_2 . First we write V'_2 as:

$$\begin{aligned}
& \delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \left(R_i \left(\underline{L} + \frac{2}{t - \underline{u}} \right) R_i^{\alpha' - 1} T^l \Delta \mu \right) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& + \delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \left({}^{(R_i)} \underline{Z} R_i^{\alpha' - 1} T^l \Delta \mu \right) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} := V'_{21} + V'_{22}
\end{aligned}$$

By the estimate:

$$|{}^{(R_i)} \underline{Z}| \leq C\delta$$

V'_{22} has the identical structure with V'_3 , therefore is a lower order term compared to V'_1 .

While for V'_{21} , integrating by parts, we have:

$$\begin{aligned}
V'_{21} &= -\delta^{2l+2} \int_{W_{\underline{u}}^t} \left((R_i T\psi) + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\pi} \right) \left(\left(\underline{L} + \frac{2}{t - \underline{u}} \right) R_i^{\alpha' - 1} T^l \Delta \mu \right) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&- \delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi) \left(\left(\underline{L} + \frac{2}{t - \underline{u}} \right) R_i^{\alpha' - 1} T^l \Delta \mu \right) (R_i^{\alpha' + 1} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} := -V'_{211} - V'_{212}
\end{aligned}$$

By the estimates:

$$|T\psi| \leq C\delta^{-1/2}, \quad |R_i T\psi + \frac{1}{2} {}^{(R_i)} \text{tr} \tilde{\pi}| \leq C\delta^{-1/2}$$

as well as Lemma 7.3, V'_{211} is a lower order term compared to V'_{212} . To estimate V'_{212} , we first consider:

$$\delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2}{t - \underline{u}} (T\psi) (R_i^{\alpha' - 1} T^l \Delta \mu) (R_i^{\alpha' + 1} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \tag{9.37}$$

By proposition 7.5, this is bounded by:

$$\begin{aligned}
& C \int_{-r_0}^t \delta \left(\mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right) \\
& \cdot \left(\mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right) dt'
\end{aligned}$$

Again, we consider the “shock part” and the “non-shock part” separately. When $t' \in [-r_0, t_0]$, the above integrals are bounded by:

$$C \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') (\delta^2 \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u})) dt'$$

While when $t' \in [t_0, t]$, the above integrals are bounded by:

$$\begin{aligned} & C\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') dt' + C\delta^2 \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2}}(t') dt' \\ & \leq \frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{C}{(2b_{|\alpha|+2}-1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \delta^2 \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned}$$

Therefore the spacetime integral (9.37) is bounded by:

$$\begin{aligned} & \frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{C}{(2b_{|\alpha|+2}-1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \delta^2 \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ & + C \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') (\delta^2 \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u})) dt' \end{aligned} \quad (9.38)$$

We end this section by estimating the spacetime integral:

$$\delta^{2l+2} \int_{W_{\underline{u}}^t} (T\psi)(\underline{L}R_i^{\alpha'-1}T^l\Delta\mu)(R_i^{\alpha'+1}T^{l+1}\psi) dt' d\underline{u}' d\mu_{\tilde{g}} \quad (9.39)$$

which is bounded by:

$$C\delta^{-1/2} \int_{-r_0}^t \delta^{l+1} \|\underline{L}R_i^{\alpha'-1}T^l\Delta\mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \delta^{l+1} \|R_i^{\alpha'+1}T^{l+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'$$

By the propagation equation:

$$\underline{L}\mu = m + \mu e$$

we have:

$$\underline{L}R_i^{\alpha'-1}T^l\Delta\mu = -\frac{dc^2}{d\rho} \psi_0 T R_i^{\alpha'-1} T^l \Delta\psi_0 + \mu \psi_0 \underline{L}R_i^{\alpha'-1} T^l \Delta\psi_0 + \text{l.o.t.}$$

Here the lower order terms l.o.t. can be bounded in the same fashion as (9.38). Now we are going to bound $\delta^{l+1} \|T R_i^{\alpha'-1} T^l \Delta\psi_0\|_{L^2(\Sigma_{t'}^{\underline{u}})}$ in terms of $\sqrt{E_{\leq|\alpha|+2}}(t')$ and $\delta^{l+1} \|\mu \underline{L}R_i^{\alpha'-1} T^l \Delta\psi_0\|_{L^2(\Sigma_{t'}^{\underline{u}})}$ in terms of $\sqrt{E_{\leq|\alpha|+2}}(t')$, so the latter is a lower order term with respect to the behavior of δ . We only estimate the contribution from the former one.

In view of the estimate:

$$|\psi_0| \leq C\delta^{1/2}$$

This contribution to (9.39) is bounded by:

$$\begin{aligned} & C \int_{-r_0}^t \delta \sqrt{E_{\leq|\alpha|+2}}(t', \underline{u}) \mu_m^{-1/2}(t') \sqrt{E_{\leq|\alpha|+2}}(t', \underline{u}) dt' \\ & \leq C \int_{-r_0}^t \mu_m^{-2a-1/2}(t') \delta \sqrt{\tilde{E}_{\leq|\alpha|+2}}(t', \underline{u}) \sqrt{\tilde{E}_{\leq|\alpha|+2}}(t', \underline{u}) dt' \end{aligned}$$

Again, considering the “shock part” and “non-shock part” separately, we have the finally estimates for (9.39):

$$\begin{aligned} & C \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') (\delta^2 \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u})) dt' \\ & + \frac{C}{(2b_{|\alpha|+2}-1/2)} \mu_m^{-2b_{|\alpha|+2}+1/2}(t) (\delta^2 \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})) \end{aligned} \quad (9.40)$$

This completes the error estimates for the top order optical terms. As we mentioned at the beginning of this subsection, although we only considered the variations $R_i^{\alpha+1}\psi$ and $R_i^{\alpha'}T^{l+1}\psi$ with $|\alpha'| + l = |\alpha|$, all the estimates in this subsection are also true for the variations $Z_i^{\alpha+1}\psi$ and $Z_i^{\alpha'}T^{l+1}\psi$ respectively. Here Z_i is either R_i or Q .

10. TOP ORDER ENERGY ESTIMATES

With the estimates for the contributions from top order optical terms as well as lower order optical terms, we are ready to complete the top order energy estimates, namely, the energy estimates for the variations of order up to $|\alpha| + 2$. As we have pointed out, we allow the top order energies to blow up as shocks form. So in this section, we shall prove that the modified energies $\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})$, $\tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u})$ and $\tilde{\underline{F}}_{\leq|\alpha|+2}(t, \underline{u})$, $\tilde{F}_{\leq|\alpha|+2}(t, \underline{u})$ are bounded by initial data. Therefore we obtain a rate for the possible blow up of the top order energies.

10.1. Estimates associated to K_1 . We start with the energy inequality for $Z_i^{\alpha'+1}\psi$ as we obtained in Section 6. Here Z_i is any one of R_i , Q and T .

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'+1}\psi](t, \underline{u}) + \underline{F}[Z_i^{\alpha'+1}\psi](t, \underline{u}) + K[Z_i^{\alpha'+1}\psi](t, \underline{u}) \right) \\ & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{1,|\alpha'|+2} \end{aligned}$$

where l' is the number of T s' appearing in the string of $Z_i^{\alpha'}$. In the spacetime integral $\int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{1,|\alpha'|+2}$ we have the contributions from the deformation tensor of K_1 , which have been investigated in Section 6. Actually, if we choose N_{top} to be large enough, then we can bound $\|\not{d}Z_i^\beta \mu\|_{L^\infty(\Sigma_t^\#)}$ in terms of initial data by using the same argument as in Section 4.2 for $|\beta| \leq N_\infty + 1$.

Another contribution of the spacetime integral $\int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{1,|\alpha'|+2}$ comes from $\int_{W_{\underline{u}}^t} \frac{1}{c} \tilde{\rho}_{|\alpha'|+2} \cdot \underline{L}Z_i^{\alpha'+1}\psi$, namely, the deformation tensor of commutators, which has been studied intensively in the last section. We first consider the lower order optical contributions, which are bounded by (See (9.14)):

$$\begin{aligned} & C \int_{-r_0}^t \delta^2 E_{\leq|\alpha|+2}(t', \underline{u}) dt' + C \delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' + C \delta^{1/2} K_{\leq|\alpha|+2}(t, \underline{u}) \\ & \leq C \mu_m^{-2b_{|\alpha|+2}}(t) \left(\int_{-r_0}^t \delta^2 \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta^{-1/2} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta^{1/2} \tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) \right). \end{aligned} \tag{10.1}$$

Here we define the following non-decreasing quantity in t :

$$\tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) := \sup_{t' \in [-r_0, t]} \{ \mu_m^{2b_{|\alpha|+2}}(t') K_{\leq|\alpha|+2}(t', \underline{u}) \}$$

By (9.28), (9.29), (9.30), (9.33), (9.31), (9.32), (9.35), (9.36), (9.38) and (9.40) the contribution of top order optical terms are bounded as (provided that δ is sufficiently small.):

$$\begin{aligned}
& \frac{C}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + C_\epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& + \epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{C\delta^{-1}}{(2b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + C\delta \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2}}(t') \left(\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \delta^{-2} \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt' \\
& + C\delta \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + C\mu_m^{-2b_{|\alpha|+2}+1/2}(t) \left(\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \right) \\
& + \frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{C\delta}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})
\end{aligned} \tag{10.2}$$

Substituting these contributions into the energy inequality, and use the fact that $\mu_m(t) \leq 1$, we obtain:

$$\begin{aligned}
& \mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'+1}\psi](t, \underline{u}) + \underline{F}[Z_i^{\alpha'+1}\psi](t, \underline{u}) + K[Z_i^{\alpha'+1}\psi](t, \underline{u}) \right) \\
& \leq C\mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \frac{C}{(b_{|\alpha|+2} - 1/2)} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\
& + C_\epsilon \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \epsilon \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{C\delta^{-1}}{(2b_{|\alpha|+2} - 1)} \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + C\delta \int_{-r_0}^t \left(\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \delta^{-2} \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt' + C\delta \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\
& + C \left(\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \right) + \frac{C}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\
& + \frac{C\delta}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + C\delta \tilde{K}_{\leq|\alpha|+2}(t, \underline{u})
\end{aligned}$$

Now the right hand side of the above inequality is non-decreasing in t , so the above inequality is also valid if we replace “ t ” by any $t' \in [-r_0, t]$ on the left hand side:

$$\begin{aligned}
& \mu_m^{2b_{|\alpha|+2}}(t') \sum_{|\alpha'| \leq |\alpha|} \left(\underline{E}[Z_i^{\alpha'+1}\psi](t', \underline{u}) + \underline{F}[Z_i^{\alpha'+1}\psi](t', \underline{u}) + K[Z_i^{\alpha'+1}\psi](t', \underline{u}) \right) \\
& \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \frac{C}{(b_{|\alpha|+2} - 1/2)} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \\
& + C \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \epsilon \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \frac{C\delta^{-1}}{(2b_{|\alpha|+2} - 1)} \int_0^u \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + C\delta \int_{-r_0}^t \left(\tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) + \delta^{-2} \int_0^u \tilde{F}_{\leq |\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt' + C\delta \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \\
& + C \left(\delta^{3/2} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \delta^{3/2} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \right) + \frac{C}{2b_{|\alpha|+2}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \\
& + \frac{C\delta}{2b_{|\alpha|+2}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + C\delta \tilde{K}_{\leq |\alpha|+2}(t, \underline{u})
\end{aligned}$$

For each term in the sum on the left hand side of above inequality, we keep it on the left hand side and drop all the other terms. Then taking supremum of the term we kept with respect to $t' \in [-r_0, t]$. Repeat this process for all the terms on the left hand side, we finally obtain:

$$\begin{aligned}
& \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \frac{C}{(b_{|\alpha|+2} - 1/2)} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \\
& + C_\epsilon \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \epsilon \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \frac{C\delta^{-1}}{(2b_{|\alpha|+2} - 1)} \int_0^u \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + C\delta \int_{-r_0}^t \left(\tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) + \delta^{-2} \int_0^u \tilde{F}_{\leq |\alpha|+2}(t', \underline{u}') d\underline{u}' \right) dt' + C\delta \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \\
& + C \left(\delta^{3/2} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \delta^{3/2} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \right) \\
& + \boxed{\frac{C}{2b_{|\alpha|+2}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} + \frac{C\delta}{2b_{|\alpha|+2}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + C\delta \tilde{K}_{\leq |\alpha|+2}(t, \underline{u})
\end{aligned}$$

The control on the boxed term relies on Remark 9.2—since $\frac{C}{b_{|\alpha|+2}}$ is suitably small, the boxed term can be absorbed by the left hand side. So if we choose ϵ and δ small enough, we obtain:

$$\begin{aligned}
& \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C_\epsilon \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' \\
& + C\delta^{-1} \int_0^u \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' + C\delta \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u})
\end{aligned} \tag{10.3}$$

Now we only keep $\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})$ on the left hand side of (10.3)

$$\begin{aligned} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) &\leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C_\epsilon \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\ &\quad + C\delta^{-1} \int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' + C\delta \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned}$$

Then by using Gronwall we have:

$$\begin{aligned} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) &\leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C\delta^{-1} \int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\ &\quad + C\delta \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned} \quad (10.4)$$

Keeping only $\tilde{F}_{\leq|\alpha|+2}(t, \underline{u})$ on the left hand side of (10.3) and substituting the above estimates for $\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})$ gives us:

$$\begin{aligned} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) &\leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C\delta^{-1} \int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\ &\quad + C\delta \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned}$$

Then again by using Gronwall we obtain:

$$\begin{aligned} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) &\leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) \\ &\quad + C\delta \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned} \quad (10.5)$$

Note that since $0 \leq \underline{u} \leq \delta$, all the constants C on the above do not depend on δ . Since the right hand side of (10.5) is increasing in \underline{u} , substituting this in (10.4) gives us:

$$\begin{aligned} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) &\leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) \\ &\quad + C\delta \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned} \quad (10.6)$$

Substituting (10.5) and (10.6) in (10.3) gives us:

$$\begin{aligned} &\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) \\ &\leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C\delta \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + C\delta^{3/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned} \quad (10.7)$$

This completes the top order energy estimates associated to K_1 .

10.2. Estimates associated to K_0 . Now we turn to the top order energy estimates for K_0 .

We first start with the energy identity for $Z_i^{\alpha'+1}\psi$, where Z_i is any one of R_i , Q and T :

$$\begin{aligned} & \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \left(E[Z_i^{\alpha'+1}\psi](t, \underline{u}) + F[Z_i^{\alpha'+1}\psi](t, \underline{u}) \right) \\ & \leq C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C \sum_{|\alpha'|\leq|\alpha|} \int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{0,|\alpha'|+2} \end{aligned}$$

Again, l' is the number of T 's in the string of $Z_i^{\alpha'+1}$.

In the spacetime integral $\int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{0,|\alpha'|+2}$ we have the contributions from the deformation tensor of K_0 , which have been investigated in section 6 and also the contribution of the spacetime integral from $\int_{W_{\underline{u}}^t} \frac{1}{C} \tilde{\rho}_{|\alpha'|+2} \cdot L Z_i^{\alpha'+1} \psi$, namely, the deformation tensor of commutators, which has been studied intensively in the last section. We first consider the lower order optical contributions, which are bounded by (See (9.15) and (10.7) and provided that δ is sufficiently small):

$$\begin{aligned} & C \int_{-r_0}^t \delta^{1/2} E_{\leq|\alpha|+2}(t', \underline{u}) dt' + C \delta^{1/2} K_{\leq|\alpha|+2}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} E_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\ & \leq C E_{\leq|\alpha|+2}(-r_0, \underline{u}) + C \delta^{1/2} \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt'. \end{aligned} \quad (10.8)$$

By (9.18), (9.19), (10.7), (9.23) and (9.24), the top order optical contributions are bounded by (provided that δ is sufficiently small and $b_{|\alpha|+2}$ is large enough):

$$\begin{aligned} & \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ & + \frac{C}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\ & + \frac{C}{(2b_{|\alpha|+2} - 1)} \mu_m^{-2b_{|\alpha|+2}+1}(t) \left(\int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \right) \\ & + C \delta^{-1+2l'} \sum_{|\alpha'|\leq|\alpha|} E[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) \end{aligned} \quad (10.9)$$

Substituting (10.8) and (10.9) into the energy inequality, and use the fact that $\mu_m(t) \leq 1$, we obtain:

$$\begin{aligned} & \sum_{|\alpha'|\leq|\alpha|} \mu_m^{2b_{|\alpha|+2}}(t') \delta^{2l'} \left(E[Z_i^{\alpha'+1}\psi](t, \underline{u}) + F[Z_i^{\alpha'+1}\psi](t, \underline{u}) \right) \leq \\ & C \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + C \sum_{|\alpha'|\leq|\alpha|} \delta^{-1+2l'} E[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \\ & C \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{C}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \end{aligned}$$

Since the right hand side of the above is non-decreasing in t , the inequality is true if we replace t by any $t' \in [-r_0, t]$ on the left hand side:

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \mu_m^{2b_{|\alpha|+2}}(t') \left(E[Z_i^{\alpha'+1} \psi](t', \underline{u}) + F[Z_i^{\alpha'+1} \psi](t', \underline{u}) \right) \leq \\ & C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \delta^{-1+2l'} \sum_{|\alpha'| \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) \\ & \leq C \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \frac{C}{2b_{|\alpha|+2}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \end{aligned}$$

As before, taking supremum on the left hand side with respect to $t' \in [-r_0, t]$, we obtain:

$$\begin{aligned} & \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) \\ & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \delta^{-1+2l'} \sum_{|\alpha'| \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) \\ & + C \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \boxed{\frac{C}{2b_{|\alpha|+2}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \end{aligned}$$

Similarly, the control on the boxed term relies on Remark 9.2—since $\frac{C}{b_{|\alpha|+2}}$ is suitably small, the boxed term can be absorbed by the left hand side. By choosing δ sufficiently small, we have:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+2}(t) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) \\ & + C \delta^{-1+2l'} \sum_{|\alpha'| \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' \end{aligned}$$

Then keeping only $\tilde{E}_{\leq |\alpha|+2}(t)$ on the left hand side gives us:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+2}(t) & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + \\ & C \delta^{-1+2l'} \sum_{|\alpha'| \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \int_{-r_0}^t \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' \end{aligned}$$

Then using Gronwall, we have:

$$\tilde{E}_{\leq |\alpha|+2}(t) \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \delta^{-1+2l'} \sum_{|\alpha'| \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u})$$

Therefore

$$\begin{aligned} & \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) \leq \\ & C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \delta^{-1+2l'} \sum_{|\alpha'| \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) \end{aligned} \tag{10.10}$$

Now we substitute this to (10.7) for $\tilde{E}_{\leq |\alpha|+2}(t)$:

$$\begin{aligned} & \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+2}(t, \underline{u}) \\ & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'+1} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) \end{aligned} \tag{10.11}$$

If we denote:

$$\begin{aligned} \mathcal{D}_{|\alpha|+2}^{\underline{u}} := & \sum_{|\alpha'|+l' \leq |\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + \delta^{-1+2l'} \sum_{|\alpha'|+l' \leq |\alpha|} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) \\ & + \delta^{2l+2} \|F_{\alpha,l}\|_{L^2(\Sigma_{-r_0}^{\underline{u}})} + \sum_{|\alpha'|+l' \leq |\alpha|+1} \delta^{2l'} \|Z_i^{\alpha'} T^{l'} \mu\|_{L^2(\Sigma_{-r_0}^{\underline{u}})}, \quad Z_i = R_i, Q. \end{aligned}$$

we can write the final top order energy estimates as:

$$\begin{aligned} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{\underline{F}}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+2}(t, \underline{u}) &\leq C \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}) &\leq C \mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned} \quad (10.12)$$

11. DESCENT SCHEME

In the previous section, we have shown that the modified energies $\tilde{\underline{E}}_{\leq |\alpha|+2}(t), \tilde{\underline{E}}_{\leq |\alpha|+2}(t)$ for the top order variations are bounded by the initial energies $\mathcal{D}_{|\alpha|+2}^{\underline{u}}$. According to the definition, the modified energies go to zero when $\mu_m(t)$ goes to zero. This means the energy estimates obtained in the last section are not sufficient for us to close the argument when shock forms. However, based on those estimates, we shall show in this section, that if the order of derivative decreases, the power of $\mu_m(t)$ needed in the definition of modified energies also decreases. The key point is that after several steps, this power could be zero and finally we can bound the energies without any weights.

11.1. Next-to-top order error estimates. We first investigate the estimates associated to K_1 . To improve the energy estimates for the next-to-the-top variations, we consider the spacetime integral (Keep in mind that the top order quantities are of order $|\alpha| + 2$):

$$\begin{aligned} & \int_{W_{\underline{u}}^t} |T\psi| |Z_i^\alpha \text{tr} \underline{\chi}'| |\underline{L} Z_i^\alpha \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \leq C \delta^{-1/2} \int_{W_{\underline{u}}^t} |Z_i^\alpha \text{tr} \underline{\chi}'| |\underline{L} Z_i^\alpha \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \leq C \delta^{-1/2} \left(\int_{W_{\underline{u}}^t} |Z_i^\alpha \text{tr} \underline{\chi}'|^2 dt' d\underline{u}' d\mu_{\tilde{g}} \right)^{1/2} \cdot \left(\int_{W_{\underline{u}}^t} |\underline{L} Z_i^\alpha \psi|^2 dt' d\underline{u}' d\mu_{\tilde{g}} \right)^{1/2} \\ & \leq C \delta^{-1/2} \left(\int_{-r_0}^t \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \right)^{1/2} \cdot \left(\int_0^{\underline{u}} \underline{F}[Z_i^\alpha \psi](t, \underline{u}') d\underline{u}' \right)^{1/2} \end{aligned} \quad (11.1)$$

Throughout this subsection, Z_i is either R_i or Q . By Proposition 7.4:

$$\begin{aligned} \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} &\leq C \delta^{1/2} \int_{-r_0}^t \mu_m^{-1/2}(t') \sqrt{\underline{E}_{|\alpha|+2}(t', \underline{u})} dt' \\ &\leq C \delta^{1/2} \int_{-r_0}^t \mu_m^{-1/2-b_{|\alpha|+2}}(t') \sqrt{\tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ &\leq C \delta^{1/2} \sqrt{\tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \\ &\leq C \delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \end{aligned}$$

Then by the top order energy estimates obtained in the last section, the integral in the first factor of (11.1) is bounded by:

$$\begin{aligned} C\delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' &\leq C\delta \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') dt' \\ &\leq C\delta^2 \mu_m^{-2b_{|\alpha|+2}+2}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned}$$

On the other hand, the second factor in (11.1) is bounded by:

$$\int_0^{\underline{u}} \underline{F}[Z_i^\alpha \psi](t, \underline{u}') d\underline{u}' \leq \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \sup_{t' \in [-r_0, t]} \{ \mu_m^{2b_{|\alpha|+1}}(t') \underline{F}[Z_i^\alpha \psi](t', \underline{u}') \} d\underline{u}'$$

where $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$. Therefore (11.1) is bounded by:

$$\begin{aligned} &C\delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}'} \\ &\leq C\delta^2 \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\delta^{-1} \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \end{aligned} \quad (11.2)$$

Next we consider the spacetime integral:

$$\begin{aligned} &\delta^{2l'+2} \int_{W_{\underline{u}}^t} |T\psi| |Z_i^{\alpha'} T^{l'} \Delta \mu| |\underline{L} Z_i^{\alpha'} T^{l'+1} \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \\ &\leq C\delta^{-1/2} \left(\int_{-r_0}^t \delta^{l'+1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \right)^{1/2} \left(\int_0^{\underline{u}} \delta^{l'+1} \underline{F}[Z_i^{\alpha'} T^{l'+1} \psi](t, \underline{u}') d\underline{u}' \right)^{1/2} \end{aligned} \quad (11.3)$$

with $|\alpha'| + l' \leq |\alpha| - 1$.

By Proposition 7.5:

$$\begin{aligned} &\delta^{l'+1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_t^{\underline{u}})} \\ &\leq C\delta^{1/2} \int_{-r_0}^t \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-1/2}(t') \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\leq C\delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\leq C\delta^{1/2} \left(\sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t') dt' + \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \right) \\ &\leq C\delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + C\delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \end{aligned}$$

Then by the top order energy estimates obtained in the last section, the integral in the first factor of (11.3) is bounded by $(\mu_m(t) \leq 1)$:

$$\begin{aligned} &C\delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') \left(\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \underline{E}_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' \\ &\leq C\delta \left(\tilde{E}_{\leq|\alpha|+2}(t) + \underline{E}_{\leq|\alpha|+2}(t, \underline{u}) \right) \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') dt' \\ &\leq C\delta \mu_m^{-2b_{|\alpha|+2}+2}(t) \left(\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \underline{E}_{\leq|\alpha|+2}(t, \underline{u}) \right) \\ &\leq C\delta \mu_m^{-2b_{|\alpha|+2}+2}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned}$$

Then again, with $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$, the spacetime integral (11.3) is bounded by:

$$\begin{aligned} & C\mu_m^{-2b_{|\alpha|+1}}(t)\sqrt{\mathcal{D}_{|\alpha|+2}^u}\sqrt{\int_0^u \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}')d\underline{u}'} \\ & \leq C\delta\mu_m^{-2b_{|\alpha|+1}}(t)\mathcal{D}_{|\alpha|+2}^u + C\delta^{-1}\mu_m^{-2b_{|\alpha|+1}}(t)\int_0^u \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}')d\underline{u}' \end{aligned} \quad (11.4)$$

We proceed to consider the spacetime error integral associated to K_0 . We first consider the spacetime integral:

$$\begin{aligned} & \int_{W_{\underline{u}}^t} |T\psi| \|Z_i^\alpha \text{tr} \underline{\chi}'\| \|LZ_i^\alpha \psi\| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \leq C\delta^{-1/2} \int_{-r_0}^t \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LZ_i^\alpha \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \end{aligned}$$

Substituting the estimates:

$$\begin{aligned} \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} & \leq C\delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \\ & \leq C\delta \mu_m^{-b_{|\alpha|+2}+1/2}(t') \sqrt{\mathcal{D}_{|\alpha|+2}^u}, \\ \|LZ_i^\alpha \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} & \leq C\mu_m^{-b_{|\alpha|+1}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+1}(t', \underline{u})} \end{aligned}$$

with $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$, and using the fact that $\tilde{E}_{\leq|\alpha|+1}(t)$ are non-decreasing in t , we see that the spacetime integral is bounded by ($\mu_m(t) \leq 1$):

$$\begin{aligned} & C\delta^{1/2} \sqrt{\mathcal{D}_{|\alpha|+2}^u} \sqrt{\tilde{E}_{\leq|\alpha|+1}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+1}-1/2}(t') dt' \\ & \leq C\delta^{1/2} \mu_m^{-2b_{|\alpha|+1}+1/2}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^u} \sqrt{\tilde{E}_{\leq|\alpha|+1}(t, \underline{u})} \\ & \leq C\mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^u + C\delta \mu_m^{-2b_{|\alpha|+1}}(t) \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}). \end{aligned} \quad (11.5)$$

Finally, we consider the spacetime integral:

$$\begin{aligned} & \delta^{2l'+2} \int_{W_{\underline{u}}^t} |Z_i^{\alpha'} T^{l'} \Delta \mu| |T\psi| \|LZ_i^{\alpha'} T^{l'+1} \psi\| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \leq C\delta^{2l'+2-1/2} \int_{-r_0}^t \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LZ_i^{\alpha'} T^{l'+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \end{aligned} \quad (11.6)$$

for $|\alpha'| + l' \leq |\alpha| - 1$. Again, substituting the estimates ($\mu_m(t) \leq 1$):

$$\delta^{l'+1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \leq C\delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t') \left(\sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \right)$$

with $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$, the same argument implies that the spacetime integral is bounded by:

$$\begin{aligned} & C\sqrt{\mathcal{D}_{|\alpha|+2}^u} \sqrt{\tilde{E}_{\leq|\alpha|+1}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+1}-1/2}(t') dt' \\ & \leq C\mu_m^{-2b_{|\alpha|+1}+1/2}(t) \sqrt{\mathcal{D}_{|\alpha|+2, l'}^u} \sqrt{\tilde{E}_{\leq|\alpha|+1}(t, \underline{u})} \\ & \leq C_\epsilon \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^u + \epsilon \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) \end{aligned} \quad (11.7)$$

Here ϵ is a small absolute positive constant.

11.2. Energy estimates for next-to-top order. Throughout this subsection Z_i could be R_i, Q and T . Now we consider the other contributions from the spacetime error integrals associated to K_1 . For the variations $Z_i^{\alpha'} \psi$ where $|\alpha'| \leq |\alpha|$, the other contributions are bounded by (see (9.14)):

$$\begin{aligned} & C\delta^2 \int_{-r_0}^t E_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + C\delta^{1/2} K_{\leq|\alpha|+1}(t, \underline{u}) \\ & \leq C\delta^2 \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+1}}(t', \underline{u}) \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' \\ & + C\delta^{-1/2} \int_0^{\underline{u}} \mu_m^{-2b_{|\alpha|+1}}(t) \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + C\delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \end{aligned} \quad (11.8)$$

In view of (11.2), (11.4), (11.8) and multiplying $\mu_m^{2b_{|\alpha|+1}}(t)$ on both sides of the energy inequality associated to K_1 for $Z_i^{\alpha'} \psi$ with $|\alpha'| \leq |\alpha|$ give us:

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \mu_m^{2b_{|\alpha|+1}}(t) \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'} \psi](t, \underline{u}) + \underline{F}[Z_i^{\alpha'} \psi](t, \underline{u}) + K[Z_i^{\alpha'} \psi](t, \underline{u}) \right) \\ & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'} \psi](-r_0, \underline{u}) + C\delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \\ & + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta^{1/2} \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) + C\delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned}$$

Since the right hand side of the above is non-decreasing in t , the above inequality is still true if we substitute t by any $t' \in [-r_0, t]$ on the left hand side:

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \mu_m^{2b_{|\alpha|+1}}(t') \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'} \psi](t', \underline{u}) + \underline{F}[Z_i^{\alpha'} \psi](t', \underline{u}) + K[Z_i^{\alpha'} \psi](t', \underline{u}) \right) \\ & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'} \psi](-r_0, \underline{u}) + C\delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \\ & + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\delta^{1/2} \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \end{aligned}$$

As in the previous section, taking the supremum with respect to $t' \in [-r_0, t]$ we obtain:

$$\begin{aligned} & \tilde{\underline{E}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \\ & \leq C\delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \\ & + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta^{1/2} \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \end{aligned}$$

Choosing δ sufficiently small, we have:

$$\begin{aligned} & \tilde{\underline{E}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ & + C\delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' \end{aligned}$$

Keeping only $\tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u})$ and we have:

$$\tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}'$$

By using Gronwall, we obtain:

$$\tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\delta\mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt'$$

This together with the fact that $\tilde{E}_{\leq|\alpha|+1}(t), \mathcal{D}_{|\alpha|+2}^{\underline{u}}$ are non-decreasing in \underline{u} implies:

$$\tilde{E}_{\leq|\alpha|+1}(t) + \tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\delta\mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\delta^2 \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' \quad (11.9)$$

Next we consider the energy estimates associated to K_0 . We start with the variation $Z_i^{\alpha'}\psi$ with $|\alpha'| \leq |\alpha|$. The other contributions to the spacetime error integral is bounded by (see (9.15) and (11.9)):

$$\begin{aligned} & C\delta^{1/2} \int_{-r_0}^t E_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta^{1/2} K_{\leq|\alpha|+1}(t, \underline{u}) + C\delta^{-1/2} \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \\ & \leq C\delta^{1/2} \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+1}}(t') \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' + C\delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned} \quad (11.10)$$

Without loss of generality, we can choose $\epsilon \geq \delta$. Then in view of this and (11.5) and (11.7), we have the following energy inequality:

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \mu_m^{2b_{|\alpha|+1}}(t) \delta^{2l'} \left(E[Z_i^{\alpha'}\psi](t, \underline{u}) + F[Z_i^{\alpha'}\psi](t, \underline{u}) \right) \\ & \leq C\mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\epsilon \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + C\delta^{1/2} \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' \end{aligned}$$

Then similar as before, substituting t by $t' \in [-r_0, t]$ on the left hand side and taking the supremum with respect to $t' \in [-r_0, t]$, we obtain:

$$\begin{aligned} & \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) \\ & \leq C\mathcal{D}_{|\alpha|+2}^{\underline{u}} + C\epsilon \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + C\delta^{1/2} \int_{-r_0}^t \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' \end{aligned}$$

Choosing ϵ sufficiently small and using Gronwall, we finally have:

$$\tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\mathcal{D}_{|\alpha|+2}^{\underline{u}} \quad (11.11)$$

Now substituting (11.11) to the right hand side of (11.9), we have:

$$\tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\delta\mathcal{D}_{|\alpha|+2}^{\underline{u}}$$

Summarizing, we have:

$$\begin{aligned} & \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\delta\mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ & \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+1}(t, \underline{u}) \leq C\mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned} \quad (11.12)$$

11.3. Descent scheme. We proceed in this way taking at the n th step:

$$b_{|\alpha|+2-n} = b_{|\alpha|+2} - n, \quad b_{|\alpha|+1-n} = b_{|\alpha|+2} - n - 1$$

in the role of $b_{|\alpha|+2}$ and $b_{|\alpha|+1}$ respectively, the argument beginning in the paragraph containing (11.1) and concluding with (11.12) being step 0. The n th step is exactly the same as the 0th step as above, as long as $b_{|\alpha|+1-n} > 0$, that is, as long as $n \leq [b_{|\alpha|+2}] - 1$. Here we choose $b_{|\alpha|+2}$ as:

$$b_{|\alpha|+2} = [b_{|\alpha|+2}] + \frac{3}{4}$$

where $[b_{|\alpha|+2}]$ is the integer part of $b_{|\alpha|+2}$. For each of such n , we need to estimate the integrals:

$$\int_{-r_0}^t \mu_m^{-b_{|\alpha|+2-n}-1/2}(t') dt', \quad \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2-n}+1}(t') dt'$$

As in the last section, we consider two different cases: $t' \in [-r_0, t_0]$ and $t' \in [t_0, t]$, where $\mu_m(t_0) = \frac{1}{10}$. If $t' \in [-r_0, t_0]$, we have:

$$\begin{aligned} \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2-n}-1/2}(t') dt' &\leq C \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2-n}+1/2}(t') dt' \leq C \mu_m^{-b_{|\alpha|+2-n}+1/2}(t) \\ \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2-n}+1}(t') dt' &\leq C \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2-n}+2}(t') dt' \leq C \mu_m^{-2b_{|\alpha|+2-n}+2}(t) \end{aligned}$$

Here we have used the fact that $\mu_m(t') \geq \frac{1}{10}$ for $t' \in [-r_0, t_0]$. In regard to the estimate for $t' \in [t_0, t]$, since

$$b_{|\alpha|+2-n} = [b_{|\alpha|+2-n}] + \frac{3}{4} \geq 1 + \frac{3}{4} = \frac{7}{4},$$

by Lemma 8.1, we have:

$$\begin{aligned} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2-n}-1/2}(t') dt' &\leq C \mu_m^{-b_{|\alpha|+2-n}+1/2}(t) \\ \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2-n}+1}(t') dt' &\leq C \mu_m^{-2b_{|\alpha|+2-n}+2}(t) \end{aligned}$$

So indeed, we can repeat the process of 0th step for $n = 1, \dots, [b_{|\alpha|+2}] - 1$. Therefore we have the following estimates:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+1-n}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-n}(t, \underline{u}) &\leq C \mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ \tilde{E}_{\leq |\alpha|+1-n}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-n}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+1-n}(t, \underline{u}) &\leq C \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} \end{aligned} \quad (11.13)$$

We now consider the final step $n = [b_{|\alpha|+2}]$. In this case we have $b_{|\alpha|+2-n} = \frac{3}{4}$. Using the same process as in 0th step, the contributions of the optical terms are bounded by:

$$\begin{aligned} \|Z^{\alpha'} \text{tr} \chi'\|_{L^2(\Sigma_t^{\underline{u}})} &\leq C \delta \mu_m^{-1/4}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \quad \text{with } |\alpha'| + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}] \\ \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_t^{\underline{u}})} &\leq C \delta^{1/2} \mu_m^{-1/4}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \quad \text{with } |\alpha'| + l' + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}] \end{aligned}$$

with $Z_i = R_i$ or Q . As before, in order to bound the corresponding integrals:

$$\int_{-r_0}^t \|Z_i^{\alpha'} \text{tr} \chi'\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt', \quad \int_{-r_0}^t \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt'$$

we need to consider the integral:

$$\int_{-r_0}^t \mu_m^{-1/2}(t') dt' \leq \int_{-r_0}^{t_0} \mu_m^{-1/2}(t') dt' + \int_{t_0}^t \mu_m^{-1/2}(t') dt' \quad \text{with } \mu_m(t_0) = \frac{1}{10}$$

For the “non-shock part” $\int_{-r_0}^{t_0}$, since $\mu_m(t_0) \geq \frac{1}{10}$,

$$\int_{-r_0}^{t_0} \mu_m^{-1/2}(t') dt' \leq C$$

For the “shock part $\int_{t_0}^t$ ”, as in the proof for Lemma 8.1,

$$\int_{t_0}^t \mu_m^{-1/2}(t') dt' \leq C \mu_m^{1/2}(t) \leq C$$

So we have the following bounds:

$$\begin{aligned} \left(\int_{-r_0}^t \|Z^{\alpha'} \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^u)}^2 dt' \right)^{1/2} &\leq C \delta \sqrt{\mathcal{D}_{|\alpha|+2}^u} \quad \text{with} \quad |\alpha'| + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}] \\ \left(\int_{-r_0}^t \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^u)}^2 dt' \right)^{1/2} &\leq C \delta^{1/2} \sqrt{\mathcal{D}_{|\alpha|+2}^u} \quad \text{with} \quad |\alpha'| + l' + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}] \end{aligned}$$

Therefore we can set:

$$b_{|\alpha|+1-n} = b_{|\alpha|+1-[b_{|\alpha|+2}]} = 0$$

in this step. Then we can proceed exactly the same as in the preceding steps. We thus arrive at the estimates:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &\leq C \mathcal{D}_{|\alpha|+2}^u \\ \tilde{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &\leq C \delta \mathcal{D}_{|\alpha|+2}^u \end{aligned} \quad (11.14)$$

These are the desired estimates, because from the definitions:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ \underline{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ E_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ \underline{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ F_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{K}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ K_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \end{aligned} \quad (11.15)$$

the weight $\mu_m(t')$ has been eliminated.

12. COMPLETION OF PROOF

Let us define:

$$\mathcal{S}_2[\phi] := \int_{S_{t, \underline{u}}} \left(|\phi|^2 + |R_{i_1} \phi|^2 + |R_{i_1} R_{i_2} \phi|^2 \right) d\mu_g$$

And also let us denote by $\mathcal{S}_n(t, \underline{u})$ the integral on $S_{t, \underline{u}}$ (with respect to $d\mu_g$) of the sum of the square of all the variation $\psi = \delta^{l'} Z_i^{\alpha'} \psi_\gamma$ up to order $|\alpha| + 1 - [b_{|\alpha|+2}]$, where l' is the number of T 's in the string of $Z_i^{\alpha'}$ and $\gamma = 0, 1, 2, 3$. Then by Lemma 7.3 we have:

$$\mathcal{S}_{|\alpha|-[b_{|\alpha|+2}]}(t, \underline{u}) \leq C \delta \left(E_{|\alpha|+1-[b_{|\alpha|+2}]}(t) + \underline{E}_{|\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) \right) \quad \text{for all} \quad (t, \underline{u}) \in [-2, t^*) \times [0, \delta].$$

Hence, in view of (11.14) and (11.15),

$$\mathcal{S}_{|\alpha|-[b_{|\alpha|+2}]}(t, \underline{u}) \leq C \delta \mathcal{D}_{|\alpha|+2}^u \quad \text{for all} \quad (t, \underline{u}) \in [-2, t^*) \times [0, \delta]. \quad (12.1)$$

Then for any variations ψ of order up to $|\alpha| - 2 - [b_{|\alpha|+2}]$ we have:

$$\mathcal{S}_2[\psi] \leq \mathcal{S}_{|\alpha|-[b_{|\alpha|+2}]}(t, \underline{u}) \quad (12.2)$$

Then by the Sobolev inequality introduced in (3.41), (12.1) and (12.2), we have:

$$\delta^{l'} \sup_{S_{t,\underline{u}}} |Z_i^{\alpha'} \psi_\alpha| = \sup_{S_{t,\underline{u}}} |\psi| \leq C \delta^{1/2} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \leq C_0 \delta^{1/2} \quad (12.3)$$

where C_0 depends on the initial energy $\mathcal{D}_{|\alpha|+2}^{\underline{u}}$, the constant in the isoperimetric inequality and the constant in Lemma 7.3 as well as the constants in (11.14), which are absolute constants.

If we choose $|\alpha|$ large enough such that

$$\left\lceil \frac{|\alpha| + 1}{2} \right\rceil + 3 \leq |\alpha| - 2 - [b_{|\alpha|+2}]$$

then (12.3) recovers the bootstrap assumption (B.1) for $(t, \underline{u}) \in [-2, t^*) \times [0, \delta]$.

To complete the proof of Theorem 3.1, it remains to show that the smooth solution exists for $t \in [-2, s^*)$, i.e. $t^* = s^*$. More precisely, we will prove that either $\mu_m(t^*) = 0$ if shock forms before $t = -1$ or otherwise $t^* = -1$.

If $t^* < s^*$, then μ would be positive on $\Sigma_{t^*}^\delta$. In particular μ has a positive lower bound on $\Sigma_{t^*}^\delta$. Therefore by Remark 2.4, the Jacobian Δ of the transformation from optical coordinates to rectangular coordinates has a positive lower bound on $\Sigma_{t^*}^\delta$. This implies that the inverse transformation from rectangular coordinates to optical coordinates is regular. On the other hand, in the course of recovering bootstrap assumption we have proved that all the derivatives of the first order variations ψ_α extend smoothly in optical coordinates to $\Sigma_{t^*}^\delta$. Since the inverse transformation is regular, ψ_α also extend smoothly to $\Sigma_{t^*}^\delta$ in rectangular coordinates. Once ψ_α extend to functions of rectangular coordinates on $\Sigma_{t^*}^\delta$ belonging to some Sobolev space H^3 , then the standard local existence theorem (which is stated and proved in rectangular coordinates) applies and we obtain an extension of the solution to a development containing an extension of all null hypersurface $\underline{C}_{\underline{u}}$ for $\underline{u} \in [0, \delta]$, up to a value t_1 of t for some $t_1 > t^*$, which contradicts with the definition of t^* and therefore $t^* = s^*$. This completes the proof of Theorem 3.1 and the main theorem of the paper.

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REFERENCES

- [1] S. Alinhac, *Temps de vie des solutions r'eguli'eres des equations d'Euler compressibles axisym'etriques en dimension deux*, Invent. Math. 111 (1993), no. 3, 627-670.
- [2] S. Alinhac *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, Ann. of Math. (2) 149 (1999), no. 1, 97-127.
- [3] S. Alinhac *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II*, Acta Math. 182 (1999), no. 1, 1-23.
- [4] D. Christodoulou, *The action principle and partial differential equations*, Annals of Mathematics Studies Vol. 146, Princeton University Press, New Jersey, 2000.
- [5] D. Christodoulou, *The Formation of Shocks in 3-Dimensional Fluids*, Monographs in Mathematics, European Mathematical Soc. 2007.
- [6] D. Christodoulou, *The Formation of Black Holes in General Relativity*, Monographs in Mathematics, European Mathematical Soc. 2009.
- [7] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of Minkowski space*, Princeton Mathematical Series 41, 1993.
- [8] D. Christodoulou and S. Miao, *Compressible Flow and Euler's Equations*, (monograph, 602 pp.), Surveys in Modern Mathematics Volume 9, International Press (ISBN 9781571462978), 2014.

- [9] G.Holzegel, S.Klainerman, J.Speck and W.Wong, *Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations: An Overview*, arXiv:1407.6276v1.
- [10] F.John *Blow-up of radial solutions of $u_{tt} = c^2(u_t)\Delta u$ in three space dimensions*, Mat. Apl.Comput. 4(1985), no.1, 3-18.
- [11] F.John *Nonlinear wave equations, formation of singularities*, University Lecture Series, 2. A.M.S Providence, RI, 1990.
- [12] S.Klainerman *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. 33 (1980), no. 1, 43-101.
- [13] S.Klainerman and I.Rodnianski, *On the Formation of Trapped Surfaces*, Acta Math. 208 (2012), no. 2, 211-333.
- [14] J.Speck *Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations*, arXiv:1407.6320v1.